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SECONDARY SCHOOL-CHILDREN'S UNDERSTANDING OF RATIO AND PROPORTION

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ABSTRACT

Secondary School-children's Understanding of Ratio and Proportion.
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Existing research on the child's understanding of ratio and proportion has usually been with children in one age group or using a small number of proportion tasks. This study describes the writing of a series of problems which appear to embody the key concepts of the topic as it currently appears in the English secondary school mathematics curriculum. The problems were used first to interview 35 children; the interviews provided information for the re-writing of the items and also enabled the researcher to ascertain the methods children use when solving ratio problems. A class test composed of 27 items was given to 2,257 secondary school children in 1976 and a further 743 children in 1977. The papers were marked for both correct and specific incorrect answers. A second phase of the study was the formulation of a hierarchy of understanding in ratio and proportion. The methods used for the formation of the hierarchy are described. Details of a longitudinal study in which children were tested three times are given, these amplify the comparison between age groups resulting from the wide-scale survey.

Children seldom appear to use a teacher-taught algorithm for the solution of problems in ratio and proportion, they adapt their methods to the demands of the question. Multiplication by a fraction is used only on the hardest items, some form of addition being used in preference wherever possible. An incorrect addition strategy (enlargement by adding a-b rather than using a/b) is much in evidence. The ability to double and halve is no indication of a true understanding of ratio and very few children in the sample (representative of the normal distribution on IQ) are able to handle a ratio such as 5:3. The three age groups tested (13+, 14+, 15+) show a small but continued improvement commensurate with an increase in age.

ACKNOWLEDGEMENT

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CHAPTER ONE

Introduction

The Problem Being Investigated

For the purpose of this study ratio is regarded as the relationship a:b and proportion as the equivalence of two ratios i.e $a/b=c/d$. In the discussion of the research which follows, problems which require the use of either of these two aspects are called 'ratio' problems. Many researchers (Piaget 1967, Karplus 1975, Lovell 1961) have found evidence that the concept of ratio and proportion is very difficult for children to understand and apply in mathematical problems. Young children have an intuitive idea of enlargement and many can cope with the ratio 2:1 but a quantitive application of a:b when not 2:1 seems to be beyond the understanding of all but the brightest adolescents (Karplus 1975, Lovell 1972, Noelting 1978). There are obviously features which distinguish between the demands of 2:1 and say 5:3, probably dependent upon the methods the children naturally use in thinking about ratio and proportion.

The purpose of this study is twofold, firstly to ascertain the methods children use to solve problems involving ratio and proportion and secondly to formulate a hierarchy of understanding in the topic of ratio and proportion. Interviews with a number of children, during which they are asked to explain their reasoning as they carry out various tasks which require a proportion strategy, are used to identify methods commonly used by children. A large number of secondary school children are tested using a written test, the development of which forms part of this study. The data obtained from this wide scale survey are then used to form groups of items at different facility levels, each group being obtained by applying statistical measures of association and mathematical descriptions. The level of understanding of each child in the sample is assessed on the basis of his success on the groups of items and this information is used to test the scalability of the groups of items. Further validation includes the collection of longitudinal data over a two year period.

Other researchers have tested children on their understanding of ratio and proportion but most have limited their work by using a small number of examples (often only one). Karplus (1972 a) has investigated the progress of children over a two year period but there is little evidence of substantial longitudinal surveys, "no one has yet taken representative samples of children and traced their growth longitudinally in respect of the scheme of proportion, across many content areas" Lovell, 1970, p.143. Hence there is a real need for a more comprehensive (large scale) investigation of the child's understanding of ratio and proportion.

Mathematics Education in Britain

Mathematics is taught everyday in British primary schools and it occupies a large part of the timetable in secondary schools. It is regarded therefore as an important part of any child's education. Most children will continue to study mathematics until the statutory school leaving age of sixteen although the content of their mathematics and the qualifications with which they leave school may vary. The subject is taught both for its utility and in order to foster an appreciation of the aesthetic nature of mathematics. The first varies from generation to generation; what was for example essential for a working man in 1920 is not necessary for today's worker; the need for speed and accuracy in computation has to a large extent been replaced by the ability to use calculators and other computational devices. At one time the aesthetic nature of the subject was considered only suitable for the few who might become mathematicians, now most developments in mathematics curriculum include some attempt at instilling an appreciation of mathematics.

The popularity of the child centred approach to learning postulated by Rousseau (1712-1778) and Pestalozzi (1746-1827), then further developed by Dewey (1859-1952) has changed the emphasis in schools to the consideration of the child as a child with the needs of childhood and not just a future adult who must be filled with the information he may need as an adult. In this century the works of Piaget, although not written specifically for classroom practice have very much influenced the materials that are presented to children. The idea that a child must be ready for learning before he can assimilate what is taught, and that he develops the complexities of his knowledge just as he develops physically, infers that the sequencing of learning experiences is of paramount importance. The sequencing may be based on either the depth of understanding of the child, displayed at different ages (Piaget) or on the skills which are necessarily prerequisite to a sought skill (Ausubel and Gagne). The sequencing of computational skills is the more straight forward for often built into a computational need is a lesser computational strategy; for example to add three digit numbers one must be able to add two digit numbers.

During the late nineteen fifties and early sixties a revolution in both the teaching and the content of school mathematics took place. Topics which had never before been in the school curriculum now appeared in elementary text books; certain topics, particularly Geometry, were fundamentally different e.g. the emphasis on transformation geometry instead of Euclidean geometry in the School Mathematics Project (S.M.P.). In the field of primary mathematics Miss E. Biggs HMI had considerable influence in Britain (Schools Council 1965, Freedom to Learn 1969). She placed considerable emphasis on the child working in a laboratory setting, using his mathematics to record the results of experiments, and learning new mathematics when there was obviously a need for it because the experiment demanded the use of a technique not yet known. The work she did with teachers espoused the discovery approach, setting the child amidst concrete materials and presenting him with a problem. The influence of Dienes was manifested in the provision of multibase arithmetic blocks and logiblocs which were to be found in most schools. His theories influenced some teachers profoundly, particularly in the schools in Leicestershire. The new mathematics in the British Primary schools in the 1960s was very much grounded in the use of concrete materials and besides the Dienes materials, Cuisenaire rods and Colour Factor blocks were very evident. The Nuffield Mathematics Project (published 1967) was founded in order to draw up a mathematics curriculum for the ages five to thirteen; its suggestions were firmly based on the use of concrete materials and to a large extent discovery learning. The aim was to prepare materials for teachers and the writing team was recruited from teachers and lecturers in colleges of education. The teachers' guides provided ideas for activities in the classroom and suggestions for workcards (the teachers were expected to produce further workcards and supplement the activities). No longer were all children in a class to be doing the same mathematics at the same time, workcards were to be written for children who needed further experience in certain topics and the teacher came to expect that children of the same age would display different needs.

The first influential 'modern' books in the secondary school were Mansfield and Thompson's "Mathematics - A New Approach" (1962-66) and the Contemporary School Mathematics Series (St. Dunstan's 1966). They introduced Matrices, Vectors, Sets and Boolean Algebra as integral parts of the secondary school mathematics course. The emphasis was on the structure of mathematics as seen through different mathematical systems. The Midlands Mathematics Project (Cyril Hope, 1963-65) produced a series of texts for 'O' level pupils and another for CSE pupils; in these there was considerable emphasis on the use of vectors. The first series of books written for the School Mathematics Project (1964) were designed for the last three years of the secondary school or post Common Entrance in public schools (initially the writers were recruited largely from public schools). The SMP books later covered the entire secondary age range and still later a lettered series suitable for a CSE course was introduced. The exercises in the books were so designed that after their completion the child was in a position to state a mathematical generalisation and were thus based on the principles of discovery learning. All these new mathematics books were written not by professional textbook writers but by practising teachers. All were tried out in schools and were thus sequenced according to the demands of the mathematics being taught and based on the experience of the teachers. There was little formal evaluation of the materials once they were in general use in the schools, popularity being taken as a sign that the series was adequate for the children for whom it was written.

With the abolition of the eleven plus examination and the move to comprehensive secondary education, the problems facing the teacher of secondary school mathematics become rather different in the 1970s from what they were in the 1960s. The children now arrive in the secondary school displaying a wide range of mathematical competence, the teacher has to tailor lessons to suit many different needs. The advent of mixed ability teaching has led to the introduction of individualised learning schemes such as those of the Kent Project and SMILE. The material for these was again written by teachers.

All modern mathematics projects of the sixties emphasised that the child should understand the mathematics he was taught, the knowledge was to be used in problem solving or in a laboratory situation. It was no longer sufficient that a child could solve problems of exactly the same type as the one he had just been shown; he needed to have sufficient knowledge of mathematics to transfer his skills to unknown situations. Most teachers aim to teach for understanding but are often content when a small proportion of a class achieve this; if the rest of the class can be made competent in the short term this is often considered enough. Many teachers of mathematics particularly in the primary schools have themselves had difficulty with mathematics; 56 percent of all women entrants to colleges of education in 1972 possessed 'O' level mathematics and of all men entrants 69 percent had 'O' level mathematics (Times Educational Supplement 1973).

Following the introduction of new topics in the mathematics curriculum, topics which neither parents nor employers had themselves learned at school, together with the change in emphasis in teaching to the needs of the child and the move away from purely computational facility, the 1970s have been considerable concern expressed by the taxpayer at the state of mathematics education. Employers are requiring considerably more mathematics of their apprentices or demanding a competence in topics that they themselves think are important even though these may not be the ones used in the work for which the apprentice is being trained. The result in Britain has been the setting up of the Assessment Performance Unit to monitor standards of attainment in consecutive years, the formation of the Cockcroft Committee to look at the state of mathematics education throughout the country, and the statement by numerous pundits that children are not numerate. Teachers and mathematics advisors are naturally very concerned and many educational authorities have drawn up guidelines and assessment schedules which state a sequence of topics and the means by which the performance on each can be assessed. The drawing up of guidelines and the order in which topics or complexities of topics are introduced are based largely on the ideas and feelings of the teachers as to what a child of a particular age should be able to achieve. A comparison of guidelines across counties shows that children in certain counties 'should' be able to accomplish certain mathematical techniques five years earlier than in other counties. There is obviously a need for some objective measure of the feasibility of such sequencing, based not only on opinion or the structure of mathematics but on hard facts regarding the actual achievement of children.

The Project
"Concepts in Secondary Mathematics and Science"

In 1974 the Social Science Research Council funded the project 'Concepts in Secondary Mathematics and Science' (CSMS) for five years. Two teams were established, one to deal with mathematics, the other with science. The aim of the project was to give information to teachers and developers of curriculum on a hierarchy of understanding in mathematics and science. The mathematics team chose to attempt this task by taking topics commonly appearing in the secondary school curriculum (ten areas were considered), writing test items which were deemed to test understanding and then testing a large sample of children using these items. The items were also used as interview instruments with a number of children. The methods used by the children for the solution of the problems (particularly those which resulted in incorrect answers) were found from the interviews; these were then used to interpret the results of the large scale survey. The results of the large scale survey provided the data from which 'levels of understanding' were ascertained within each topic; the totality of levels within each topic area formed individual hierarchies. Ratio was selected for inclusion in the CSMS testing as it is a topic commonly occurring in the secondary school mathematics curriculum and it is widely used in Science.

Other researchers have investigated the strategies used by children when faced with a task requiring the application of ratio or proportion. Piaget identified the use of a proportion schema as an indication of formal operational thinking. In clinical interviews Piaget investigated proportion on a number of occasions: in the equilibrium of the balance (The Growth of Logical Thinking 1958), in the enlargement of triangles and rectangles (The Child's Conception of Space, 1967) and in the amount of food fed proportionately to eels ('Epistemologie et Psychologie de la Fonction", 1968). In each experiment he interviewed children of different ages and outlined the strategies they used for solution, assigning levels of cognitive thought to specific methods of solution. Karplus (1975) also identified different methods used by children when attempting to solve the "Mr. Short and Mr. Tall" problem. His research involved not only American children but also pupils from six European countries. The sample was restricted to one age group (13 and 14 year olds). The children interviewed by Piaget and his fellow researchers in Geneva were often of below secondary school age. In general no attempt was made by either researcher to investigate methods used by the same children on a number of different problems although the Piagetian examples sometimes provided increasing degrees of complexity in the one example. The research already reported does not give a complete picture of the child's understanding of proportion and ratio.

Purpose of the Study

As indicated previously there is a real need for a description of a hierarchy of understanding in the topic of ratio. Such a hierarchy when seen in conjunction with hierarchies in other mathematical topics also being researched by the CSMS Team should be invaluable to writers of curriculum materials and particularly for those drawing up the guidelines for teachers (already being produced by some counties). It is anticipated that the identification of such a hierarchy should enable teachers to sequence their presentation of the topic and to be aware of the difficulties experienced by children. In order to provide evidence for such a hierarchy a large number of children must be tested on problems of varying degrees of complexity. An important aspect of the research is the development of a test instrument which will later be made available to teachers.

The large scale testing should provide not only information for the development of a hierarchy but also serve to identify common errors committed by children when trying to solve problems involving ratio and proportion. The interviews with children provide information on how children attempt such problems; information which is not available from their performance on the written test. The answers to the written test are interpreted on the basis of the interviews. Any method used by children which cannot be ascribed to a teacher taught rule or algorithm is especially noted. If teachers can be made aware of the methods that children use naturally they may be able to adapt their own teaching methods to take these strategies into account.

In order to compare the levels of understanding in ratio with those in other mathematical topics all children in the CSMS sample are asked to complete two test papers. In addition 500 children besides completing a mathematics test are given a class test composed of examples taken from the works of Piaget. If the children can be described in terms of their cognitive level on the basis of this test then their level of attainment in ratio can be matched to it, in the sense that certain types of items might be described as requiring formal operational thought etc. Progress from one level of understanding to the next may be apparent in the differences in performance of the three age groups being tested. Further information on the progress of understanding as the child gets older is available from a longitudinal study in which 200 children are tested three times in two consecutive years.

Although the teachers of the children being tested are asked to state the methods and materials they use when teaching mathematics no attempt is made in this thesis to compare different teaching styles or textbooks.

Outline of the Thesis

The next chapter consists of a review of the literature on the understanding of ratio and proportion, with an emphasis on the works of Karplus and Piaget, including a discussion of those items from their work which were used in the CSMS test. In addition this chapter reviews some of the work on hierarchies in mathematics understanding. Chapter three describes in detail the construction of the test, including the steps taken prior to the formation of the test instrument; these were an analysis of the topic of ratio as it appears in commonly used secondary textbooks, an identification of the aspects of the topic for which items are written and the writing of the items. The pilot testing which took place prior to the wide scale testing is also reported in chapter three. Chapter four gives details of the implementation of the testing including the sample chosen for both the survey and the longitudinal study. A discussion of the measurement techniques available for the establishment of a hierarchy, especially those used by other researchers, also appears in this chapter.

Chapter five deals with the interviews and includes a description of the methods (both correct and incorrect) used by the children. In the large scale testing certain incorrect answers are noted, the incidence of these is reported in this chapter.

Chapter six describes the results obtained from the testing, the formation of the hierarchy and the statistical methods used together with mathematical descriptions of each level. Each child is assigned to a level of understanding and from this information the performances of the three different age groups are compared. The results from the longitudinal survey further amplifying the comparison. The hierarchies obtained in other topic areas are compared with that found in ratio. Chapter seven consists of a discussion of the results and the implications for the teaching of ratio and proportion in the secondary school.

CHAPTER TWO

Literature Related to Levels of Understanding and to the Topic of Ratio and Proportion

The study is concerned with the understanding by secondary school children of the topic of ratio and proportion. A large sample of children was tested and from the resulting data a hierarchy of understanding based on the items in the given test, was drawn up. Children were then assigned to a level of understanding on the basis of their performance on the test, not in the form of an overall score but as a measure of the type of questions at which they appear to be successful. The research relevant to the establishment of a hierarchy of understanding, particularly that which involves the use of statistical methods is quoted in this chapter. The topic of Ratio and Proportion, as understood by children has been researched by many; some of the principle researches in this area are reported in this chapter.

Ratio and Proportion

The ability to handle ratio or proportion is considered an important part of a child's mathematical attainment, not only because of its use in mathematics itself e.g. enlargement, similarity of figures, trigonometry; but also because of the widespread applications of the topic in Science. The whole area of fractions is of course concerned with the ratio of two integers but will not be dealt with in this study except in those instances where tasks involving proportion require their use. Ratio and proportion is seen by many as part of a wider aspect of mathematical knowledge, for example being indicative of the ability to handle abstractions or as part of the whole concept of a function. Lunzer (1973) saw the problem of the child's understanding of ratio as closely linked to his appreciation of relations and so functions:-

> "An alternative and equally valid interpretation sees multiplication as a function, or one-many relation. It is the latter which is relevant to direct proportionality. Moreover, it is this aspect which is essential to a correct understanding of inverse proportionality."
>
> p.13

Suarez and Biner (1978) suggested that many of the studies carried out on the child's understanding of proportion "Pay too little attention to the possibilities of favouring the ability of proportional thinking in pupils by introduction of situations in which the structure of linear function emerges." p34.

Piaget described the understanding of proportionality as an integral part of formal reasoning and much research on the topic has been allied to testing his theories. Both Piaget and Karplus have tended to use a problem which requires the use of proportion for its successful completion, they have then analysed and categorised the responses of the children. The categories were usually put in some order implying a hierarchy of responses. Besides investigations of responses to one problem there is also research where the problem itself is made successively more difficult by the introduction of more complex ratios (Noelting 1978) thus providing a hierarchy based on the values of a:b used in the one problem.

The research quoted in this chapter is primarily concerned with the work of Piaget and Karplus and the investigations they have carried out, since examples from these investigations occur in the test later described. The work is firstly described and afterwards the responses given by the children are analysed. Other research is quoted in that it attempts to validate or extend the work of Karplus and Piaget. Some research with young children and their attempts to solve problems of proportionality is quoted, since although this study deals with secondary school children the naive responses given by children of this age are closely akin to those given by young children. Finally research which deals with an analysis of what is required in the successful teaching of the topic is quoted.

In Piagetian theory there are four phases of cognitive development allied to physical growth (or age), these are the sensori-motor stage, the pre-operational stage, the concrete operational stage and the stage of formal operations; each may be subdivided. An operation may be described as:

> "A reversible, internalisable action which is bound up with others in an integrated structure...

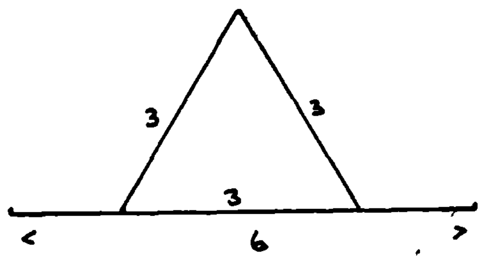
> Roughly, an operation is a means for mentally transforming data about the real world so that they can be organized and used selectively in the solution of problems. An operation differs from simple action or goal-directed behaviour in that it is internalized and reversible."
>
> (Piaget and Inhelder, 1958, ppxiii-xiv).

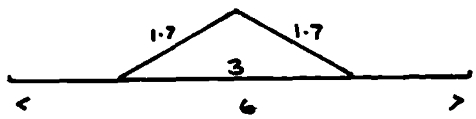
Piaget's description of the formal operational stage (age 11-12 in Geneva children) stressed the ability to handle hypotheses, the ability to control for a number of variables and the ability to deal successfully with proportion. He further elaborated on the use of the operations of formal logic by the adolescent at the formal stage, this has been open to some criticism (see Howe 1974 for a discussion of this). It is no part of this thesis to investigate the claims for the INRC system but simply to look at ratio in the light of some of Piaget's research, as stated by Lunzer (1973) :

> "It is worth noting that while an adequate mastery of the topic certainly entails an understanding of the whole set of relations.
>
> $$\frac{a}{b} = \frac{c}{d} \rightarrow ad = bc \rightarrow \frac{a}{c} = \frac{b}{d}$$
>
> together with their inverses, it has not been shown that this realisation comes about spontaneously as a result of maturing logic."
>
> p.13

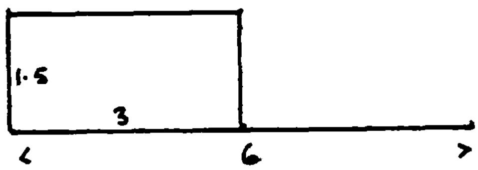
Piaget's Research on Proportion

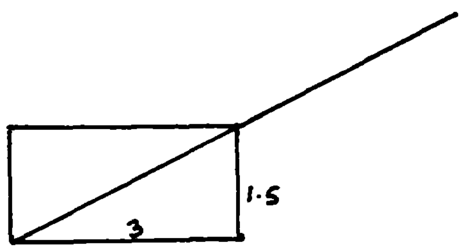
In "The Child's Conception of Space" (1967) Piaget and Inhelder investigated the problem of metric proportion with respect to the triangle and rectangle. The task which presented similar triangles was to have two outcomes, firstly to explore the parallelism of the sides as a criterion of similarity between triangles and secondly to facilitate the study of the relationship between this criterion and that of the equality of the angles compared with length of sides. The method used was to provide the child with a triangle, a new base and ask him to draw the correct triangle to circumscribe the one he was given originally. The midpoint of the new base was the same as the midpoint of the original base. The first part used only isosceles triangles, later irregular triangles were introduced. The emphasis was on the discovery of the parallelism of the sides. Diagrams showing the type of example used appear below.





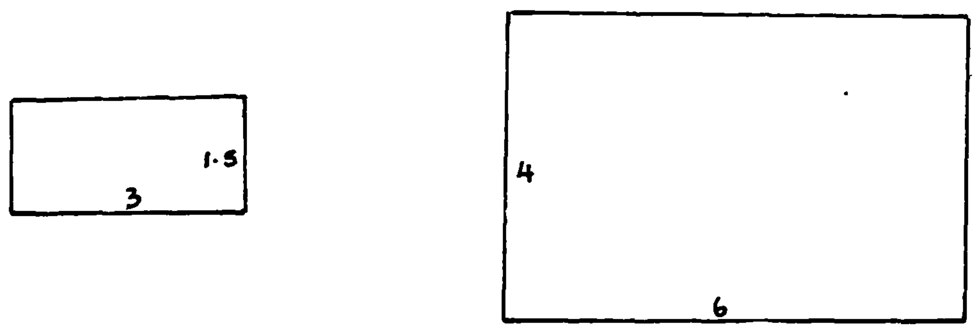
Secondly the child was presented with a triangle on which two sides had been extended and his problem was to provide just one side. The investigation was extended to examples where the base was covered, depriving the child of a line with which to draw his parallel, then he was presented with triangles already drawn and asked to draw similar ones which were not circumscribed. Lastly he was asked to sort sets of triangles into similar and non similar pairs. For comparison the child was asked to circumscribe a given rectahgle with a similar one, compare similar and non similar rectangles and draw a rectangle circumscribing the given one which had its diagonal extended (in order to see whether the child made use of the extension). See diagram below.





A further group of questions dealt with similarity of triangles when the child was allowed to handle them freely, superimpose and choose similar ones as he wished. The two types of presentation allowed the authors to categorise the children's responses in substages IIA → IV but not lower. Children performing at Substage IIA took no account of parallelism or equality of angles, at substage IIB they showed an intuitive idea of parallelism in a few cases. "Stage III marks the appearance of operations facilitating general comparison of parallels, angles and simple dimensional relations" p.326 At stage IIIB the children made comparisons which took account of both parallelism of elementary dimensional relations and when superimposing triangles they related parallel sides and equal angles. Stage IV showed the attainment of true proportionality for all dimensional relations. Judging the similarity of rectangles was more difficult than judging the similarity of triangles.

The study of similar rectangles subsequently reported was undertaken because although rectangles are less complex than triangles in that all the angles are right angles, they provide less clue to the lack of similarity (which is apparent in non similar triangles) Using phrases such as "is the same shape but bigger" Piaget presented rectangles, the smallest of which was 1.5cm x 3.0cm; the child was asked to choose those which were similar to the small one (see below).



Next the child was asked to draw a rectangle similar to the 1.5cm x 3.0cm one given.

It was impossible to define a stage for the very youngest children. Stage II children chose rectangles which were too long, this occurred as well when the child was asked to draw. They showed no desire to measure and proportions appeared meaningless. Stage III (appearing at the age 7 or 8) showed spontaneous attempts to measure but the children did not realise that proportions rather than increase in absolute size were involved and the length of the rectangle was exaggerated. During substage IIIB the figure was enlarged by the addition of a length; only in the case of the ratio 2:1 were the correct answers given. It was only at stage IV that the child began to understand proportionality and his constructive thought outweighed perception.

An experiment using an open figure, where three lengths had to be enlarged, was also reported; the stages being in the same order but the advent of each a little later. The ability to use metric proportions on all three lengths being apparent at about age eleven. It is worth noting that even at stage IV on the closed rectangles, only one child of twelve years was quoted as multiplying by a non integer rather than trying to build up to an answer by using the difference in lengths of the two figures.

In "Epistemologie et Psychologie de la Fonction" (1968) Piaget and his co-authors dealt with proportionality in other settings. The experiments were quoted by Lovell (1971) in the N.C.T.M. booklet on Piagetian research. Lovell described the experiments and stated that the findings were suggestive "but they are in great need of confirmation" p.138.

The first described was the presentation to the child of 'fish', five, ten and fifteen centimetres in length, the child was provided with firstly 'meat balls' and secondly 'biscuits' with which to feed the fish; they had appetites corresponding to their lengths. The questions asked were:

1. If fish A (5cm) gets one ball, how much must be given to B and C?
2. If B (10cm) gets four balls, how much must be given to A and C?
3. If C gets nine balls, how much must be given to A and B?

The questions were repeated using the continuous quantity of biscuits. Piaget and his fellow authors divided the children's responses into four main categories. The first or most naive being when the child reasoned that if the fish was larger he received more food but any extra amount would suffice. At the next level the child attempted to provide some regularity in the increase of the amount but gave only one more for the next largest in size. The third stage showed a more systematic increase in food, sometimes two units more for each of the larger fish: "Dir (7.4) donne 5, 7, et 9 : J'enleve 2 de 9 et j'enleve 2 de 7 ". This third stage was described by Lovell as the use of pre proportions "If the difference between A and B is a and that between B and C is b, then the child's preproportionality is of the form a is to a' as b is to b', but in which the equality of crossproducts is missing." p.139. The fourth stage was the use of metric proportions. The questions using the biscuits were on the whole harder than those using the meat balls and each stage occurred later.

Lovell also called attention to five other experiments reported in "Epistemologie et Psychologie de la Fonction", each dealing with proportionality. These were (Lovell 1971) :

1. Rectangles that have constant perimeter but in which the length of one side is decreased, leading to an increase in the adjacent side.
2. Serial regularities of diameters and positions of rings placed on rods of different lengths which were themselves set at fixed distances apart.
3. The distance travelled by a point on the rim of a wheel in relation to the size of the wheel.
4. The relationship of the size and frequency of rotation of wheels and the distances travelled by objects at the end of strings the other ends of which encircle the wheels.
5. The relationship of the magnitude of a weight and its distance from the fulcrum when the arm of the balance is in equilibrium. (quoted later from 'The Growth of Logical Thinking, Piaget 1958).

Lovell summarised the findings by describing the first stage (lasting up to about eight years of age) as an inability to coordinate variables, i.e. a decrease in height of the rectangles does not bring about an increase in width. The second stage displayed the starting point of the appreciation of "all functional variations", an awareness of simple correspondence but a lack of comparison between absolute differences and no direct or inverse compensation. The third stage arose at about ten to twelve years of age and showed the beginning of true proportionality, for example in experiment two above, it was realised that the distances between the rods must vary. It was not until stage four that the full understanding was reached, Lovell said "In Piaget's view, two conditions must be fulfilled before this stage is fully reached: the pupil must be able to both handle the boundary conditions of the variables and the ratios between the successive ordered values of the variables". p.141. The child must in fact be capable of quantifying the proportion rather than describing it in qualitative terms.

In "The Growth of Logical Thinking" (1958) Piaget dealt with the beam balance experiment where the child was asked to place weights on either side of the fulcrum so that the bar remained horizontal. An effective performance was obtained at level IIB (age 10+) but the explanation was in terms of 'the heavier it is, the closer to the middle'. It was not until levelIIIA (early formal) that the proposition was given by the child in terms of W/W' = L'/L (where W and W' are two unequal weights and L and L' the distances at which they are placed). This explanation truly belonged to stage IIIB (late formal) but sometimes appeared at stage IIIA.

Piaget generalised from other experiments dealing with proportion and further described a characteristic view of the situation shown by children at the IIB stage:

> We should remember that an understanding of proportions does not appear until substage IIIA; this is true in all spheres and not only in the balance scale experiments. During substage IIB it has often been noted that subjects search for a common denominator of the two relations that they compare, but this common relation is thought to be additive. Thus, instead of the proportion W/W' = L'/L one would have an equality of differences W-W' = L-L'. p.177

Describing the different levels of performance on the task which has a truck (containing different weights) attached to a counterweight and balanced on an inclined plane, Piaget again gave a generalised description of concrete operational behaviour:

> As long as the subject is limited to using concrete operations of classes and relations, he cannot determine the law --------

> The explanation is two fold; first the correspondences which must be empirically established are too complex, and second, the products of the multiplications of relations are in part indeterminate. p.190

He has already explained that at the concrete stage the child cannot simultaneously take the three factors (weight in truck, counterweight and slope) into consideration. The child might deal with them in pairs but forgets the third factor. At the formal stage the child seeks to co-ordinate the three factors into a single law. The proportionality statement comes about by substituting the height of the plane for its inclination and this does not appear until the late formal stage. In the interviews the substitution of height was suggested to one child who immediately afterwards formulated the law of proportionality, a second type of response came about because there was a search for proportionality and the factor of height arose as a consequence of the search. The realisation of what was needed was therefore fairly immediate in each case.

The distinctive feature of the late formal operational level on the task of placing different size rings at distances from a screen so that the shadows of two are identical, was that the children had already formulated a hypothesis:- "You have to put the largest the furthest away, and the ratio between the diameters of the rings and the distances has to be the same". The early formal replies took into account the distances from the screen of the first ring and not simply the distance between rings (as in IIB) but the children did not have a generalised hypothesis.

Generalising on the proportionality schema based on replies to the three tasks mentioned above and a fourth concerning balls of different weights placed at different distances from the centre of a spinning disc, Piaget said:

> Given two independent variables, the subject constructs the qualitative proportionality schema when he understands that an increase in one gives the same result as a decrease in the other. In all cases the structure of proportions requires an element of compensation". p.219

To find types of behaviour at each of the levels of operational thinking one might signify late formal behaviour by the appearance of the use of hypotheses, an appreciation of the multiplicative relation required and an ability to deal with three variables. At the early formal stage the child can often deal with specific situations but not state a general law. At the late concrete stage the child does see there is a relationship but looks at differences (an additive relationship rather than a multiplicative one) and further faced with three variables ignores one and copes with just two.

Research Linked to Piaget's Work

Lovell (1961) repeated some of the experiments from "The Growth of Logical Thinking" using British children and college students as subjects. Each subject was examined individually on four experiments and asked to perform certain tasks, Lovell generally confirmed the main stages of the development of logical thinking suggested by Inhelder and Piaget. He did however find that only very bright twelve year olds performed at the level of formal reasoning.

Lovell (1972) further elaborated on the lateness of the amergence of formal reasoning:

> Our work at Leeds has indicated that pupil's responses in respect to the construction of a rectangle similar but larger than a model can be placed more or less - more or less - into the categories which Piaget suggested, but the ages at which the stages are reached have been much higher.---- The studies of ourselves at Leeds (Lovell 1961, Lunzer 1965, Lovell and Butterworth 1966) with British pupils, also of Steffe and Parr (1968) Gray (1970) with American pupils, just to mention a few studies, have all confirmed that apart from very able twelve-year olds, it is from 12 years of age onwards, the actual age depending on the ability of the pupil, that facility is acquired in handling metric proportion. Many pupils may not be able to do this until 14 or 15 years of age and some never. p.8.

The study by Lovell and Butterworth (1966) distinguished between ratio and proportion defining proportion as "a relation between relations --- the child is able to recognise the equivalence of two ratios". The purpose of the study carried out with children aged 9 - 15 and using both written answers and verbal justification on the proportion and ratio tasks, was to test the following hypotheses:

1. The schema of proportion depends on some central intellective ability which underpins performance on all tasks involving proportion.
2. In addition to some central intellective ability, specific abilities contribute to the ability to use proportionality in particular tasks.
3. Tasks involving ratio will depend less on the ability indicated under a) than in the case of the tasks involving proportion.

The twenty tasks given included a non verbal reasoning test and one involving verbal analogies. A score was assigned to each reply, a score of six being allocated to the reply which showed formal reasoning where that was apposite, in the two non-mathematical tasks a score of six meant that all questions were dealt with correctly. The method of analysis was a principal component analysis producing four components all with eigen values greater than unity. The authors interpreted the loadings on the components thus: "The table clearly reveals that there is a large general component, accounting for a little over 44 per cent of the variance, and which reflects some central intellective ability embracing the schema of proportion". The question having the highest loading and which was stated to be the clearest example of the equivalence of two ratios was

"Calculation of the missing numbers:

16	12
8	6
10	$7\frac{1}{2}$
?	9"

By consideration of the component analysis the authors stated that their three hypotheses were confirmed. Finally they pointed out that it was not until 15 years of age that even fifty per cent of the responses were at the formal level.

Lunzer and Pumfrey (1966) quoted a series of experiments involving proportionality, including the shadows task from Inhelder and Piaget. Working with 25 children aged 6 to 14, Lunzer presented a wall of Cuisenaire rods of one colour and the child was asked to make a wall of the same length using a different colour. To help the subject find the correct number two procedures were used. The subject was either shown two green rods aligned with three red rods (lowest common multiple match -LCMM) or he was shown one of each variety with the correct number of white unit rods matched against each. Even the youngest children could use the first presentation, they just repeated the array. When the blocks to be used by the child were an exact multiple of the lengths of those used by the experimenter, for example 2:1, the children were able to find a matching (virtually providing their own LCMM). The other methods used were a building up using "numerical equivalence followed by addition or subtraction of difference". Some children could use a multiplicative strategy. The authors in their summing up picked out the ratios 1:1, 2:1, 3:1 as exceptional cases (being easier than other ratios). They further said:

> At all ages children seem to prefer to look for additive modes of solution even when the problem could suggest multiplicative methods. When the former are of no avail, success is not reached until well into the secondary years. p.11.

Pumfrey (1967) endeavoured to combine the aspects of Piaget's clinical method with the requirements of objective data collection in his research on proportional reasoning. His subjects were 80 children between the ages of five and fifteen (four boys and four girls at each age level). The tasks were (i) the balance problem based on Inhelder and Piaget (ii) building a wall of Cuisenaire rods, the subject was asked to predict the number of bricks of a different colour (thus different lengths) he would need to build a wall of the same overall length (iii) using a pantograph when the subject was asked to predict the direction and amplitude of the movement of one pointer of a pantograph with respect to the other. The results were analysed according to two criteria a) the accuracy of prediction b) the strategy used by the child. The balance question was found to be most difficult by most of the children, the cuisenaire rod task being the easiest. Some twenty six different strategies were used by the children in the balance task. The children's ability to produce new strategies reached a maximum at age 12 - 13. The decline in number occurred at the age when the children were beginning to make consistently correct predictions using the schema of proportionality. Thus they were beginning to function in a more effective way.

Research of Renner in Science Education

Renner (1977) investigated College students, in particular students who were freshmen in four universities (they were enrolled in an English course). He concluded that they were competent in basic arithmetic but "We do find, however, a basic deficiency in entering college freshmen when they are required to handle a ratio or proportion of any kind." p.286. Renner and his associates later tested 99 students on the concept of ratio, the items included similar triangles, rate and percentage. He found that thirty percent of those who were non-science or non-mathematics majors could not compute a simple percentage. His conclusion was that many students leave college incapable of 'formal thinking'. He presented students in a physics course with an example requiring the conversion of imperial to metric units and categorised the responses as : no response, simple arithmetic involving addition or subtraction, computation of the relationship mile : kilometre but no further application, solving the problem in two distinct steps and finally using what was essentially $\frac{a}{b} = \frac{c}{d}$. About thirty percent of the students were categorised as being at each of the last two stages. In a report of the Cognitive Analysis Project (1977b), set up to provide materials by which teachers could measure the intellectual development of a large group of children simultaneously, Renner described the response categories to two questions involving ratio. One involved the shadows of a post and a building measured at the same time of day, the building's shadow being fifty metres, the height of the post three metres and the shadow of the post two metres. The problem was to find the height of the building. A very careful analysis of the replies was made and after initially categorising responses as in the study above, the team finally decided on seven major categories. The most naive answers involved the use of irrelevent or manufactured numbers, level three gave answers such as 51, recognising that a relationship was involved and having class inclusion in mind. At level four the student recognised and stated the ratio 3:2 but did not use it. At the next level the student used the ratio but incorrectly e.g. 2/3 of 50 instead of 3/2 x 50. The last two categories demonstrated the proper use of proportion, the highest being assigned to those students who mentioned other problem variables, such as "at the same time of day". The scores on these items were correlated with the scores assigned on interviews using Piagetian tasks (EI scores), Renner concluded that the items involving proportion were the best predictors of the students' EI score or Piagetian level of reasoning.

The Research of Karplus

Karplus et al have published various findings on research dealing with children's understanding of proportion. In May 1970 Karplus and Petersen reported on research dealing with a version of the Mr. Short and Mr. Tall task which was later used in a different form in a wide scale survey. The 1970 version had two pin men each of which was measured by the experimenter demonstrating to the class of children. The first measurement was with large paper clips (biggies), Mr. Short being four biggies tall and Mr. Tall being six biggies in height. The children were then asked to measure a smaller version of Mr. Short using small paper clips (smallies). The problem was to predict the comparable height of Mr. Tall in "smallies". The replies were categorised in seven ways, the first (N) being no explanation. Category I involved intuition or guessing, category IC was assigned to children who used the data haphazardly and in an illogical way i.e. some inaccurate reasoning was present. Category A was assigned to those children who used all the data but who applied the difference rather than the ratio of measurements thus saying "the little man was 4 of his and 6 of mine so I added 2". Category S (scaling) used multiplication but not by the correct factor, 90 percent of these children in fact doubled. Category AS (addition and scaling) involved a multiplicative strategy combined with an additive one e.g. "I think two smallies are as big as one biggie, so I added four smallies for the two extra biggies." Finally there was the category (P) for children who could solve the problem by setting up a ratio. Karplus did not order these categories and found that although the children from suburban homes achieved level P by the time they left school this was not true of the urban children in his study. Two years later Karplus (1972) used the same item with the same children to find out the progress towards level P of each child and in order to hypothesise an ordering for the responses, particularly levels AS, S and A. In the original experiment about one fourth of the children reaching level P in fact used a geometric method of solution, dividing Mr. Short into fourths by visual estimation and then extending Mr. Short's height by two fourths, the extended length being measured with their chain of 'smallies'. Karplus stated that levels P and AS were more advanced than the others because 28 percent of those in other categories moved into these two over the two years, more than a third of the students showed no movement at all. Categories I and IC were regarded as the most naive types of response since 65 percent of subjects in these categories moved into others over the two years. The study did not provide evidence for the ordering of the categories A and S since the same number of children moved from A to S as moved from S to A.

Karplus (1972b) extended the research using a second version of the Mr. Short and Mr. Tall problem, this time the measuring was done with buttons and paperclips. The sample was chosen from fourth to eighth graders (age ten to fourteen years). This second version of the task did not show Mr. Tall or the buttons, Karplus concluded that on this task the children relied on the measurements and not on their perception as they had done in the previous problem. The categories of response were extended and subdivisions of the AS and P responses described in the earlier research were provided. The following categories of response remained the same: N (no explanation), I, IC, S and A. AS was now defined as an explanation focussing on the excess height of Mr. Tall and using a scaling up of the two excess buttons by a factor not mentioned in the data. AP was a similar explanation but the scaling factor for the two buttons was based on the data. PC was an explanation which used the relation that a button was about 1.5 paper clips but the relation was obtained by measuring a quarter of Mr. Short's height with paper clips (frequently shown by pencil marks on the paper). Finally category R provided an explanation where the scale factor was derived directly from the data and the factor was applied by multiplication. The results showed that category S responses were now very small in number compared with the percentage of children providing this reply on the first version of the task. An attempt was made to equate the responses to the stages used in Piagetian theory, these will be described later. Karplus also commented on the effect of changing the ratio:

> Careful examination of the responses has convinced us that a student's ability to recognise and apply a 2:1 ratio is not sufficient evidence of proportional thinking. The 3:2 ratio involved in the task we have described is significantly more demanding. Karplus (1972b) p.5.

The Mr. Short and Mr. Tall task was used with another task which involved the control of variables, (E and R Karplus in 1974);the sample being eighth graders. This work was a pilot study for a larger cross cultural study carried out in Europe. The countries visited in the seven nation survey,[Karplus et al (1975)], were Denmark, Sweden, Italy, United States, Austria, Germany and Great Britain. No claim was made by the researchers that the sample from each country was representative of the child population and each was described in terms of its socio-economic setting. The second version of the Proportional Reasoning task was used and the response categories telescoped to provide just four : I (intuitive) - not making use of all the data or using the data in a haphazard way; A(Additive) - the explanation focussed on the difference; Tr (Transitional) - the explanation showed only partial proportional reasoning or made reference to concrete comparisons; R(Ratio) - the explanation used a proportion or derived the exact scale, no concrete or iterative procedures were employed. A further proportion question was included - "Mr. Tall's car is 14 paper clips wide. How wide is Mr. Tall's car, measured in buttons?". The car question proved to be more difficult. A composite score was obtained from the pair of responses for each child, if the scores were different, category R combined with any other was called Tr, Tr combined with A was scored A, and either Tr or A combined with I was scored I. The percentages for 3,300 children aged 13.6 to 14.6 were:

Category	I	A	Tr	R
per cent	28	15	32	25

The task for assessing the students' reasoning which involved the control of variables (another concept identifying the stage of formal reasoning in Piagetian terms) described the collision of two spheres rolling on a track. After identifying the important variables the students could use the principle that only one variable at a time must be changed if an experiment is to be conclusive. The children in some countries found this very much more difficult than the proportional reasoning task.

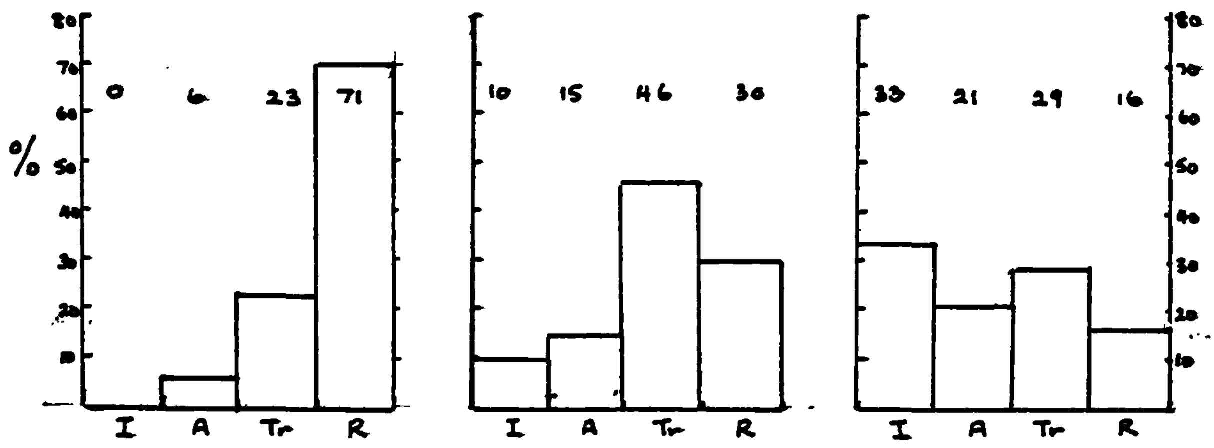
Karplus concluded that about seven percent of his total sample were formal thinkers, this was shown by the fact that they were fluent in the use of ratio and scored high marks on the control of variables task. The distribution of the responses varied from country to country and from one socio-economic group to another, in particular the incidence of the additive strategy appeared to be very low for some of the populations. Karplus therefore concluded that the additive strategy was not a part of the development of the understanding of ratio : "The present data strongly suggest that additive reasoning does not lie on an invariant development sequence but is strongly influenced by instruction and represents an effort by students to deal with a task in an ad hoc rather than a systematic way." (Karplus et al, 1975 p65)

The British sample was drawn from schools in London; Grammar schools, Independent schools and Comprehensive schools; the first two types of school being highly selective. The children were completing their third year in secondary school. The number of students from each type of school was small, sometimes two classes only. The results are shown in fig.1 below:

Figure 1. Proportional Reasoning Data, Great Britain (Karplus, 1975)

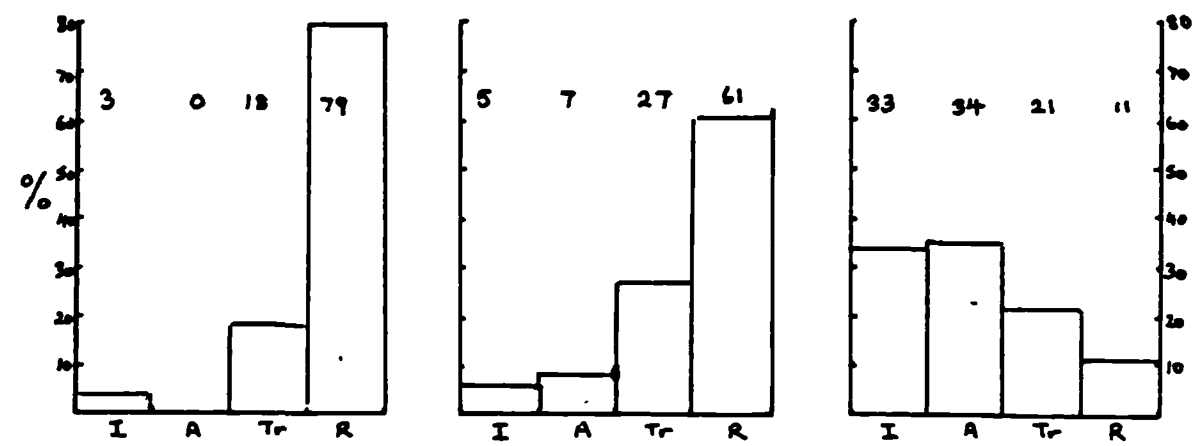
Proportional Reasoning Task, Great Britain, Male.

Direct Grant (48) Grammar (61) Comprehensive (85)



Proportional Reasoning Task, Great Britain, Female.

Direct Grant (39) Grammar (56) Comprehensive (87)



As can be seen, the response patterns differed markedly depending on the type of school from which the sample was drawn. There was a high incidence of the additive strategy in the Comprehensive school sample's responses.

An Analysis of the Responses Given to Ratio and Proportion Questions.

The highest level of attainment described by Renner was that where in addition to using a quantitative form of ratio the child was able to make statements about the conditions applying to the variables involved. This would seem to require rather more than the correct solution of the Mr. Short and Mr. Tall problem by a non-concrete strategy, however since Karplus required success on both the proportion problem and on the control of variables problem before he declared the child was at the stage of formal reasoning; the success on the proportion task only might be regarded as less than late formal reasoning. Easley and Travers (1976) pointed out that:

> They (Karplus et al) are careful not to identify what they call proportional reasoning ...... with Piaget's Formal Operations. At least, they are quite open to the possibility that these may turn out to be different things. p.39

Easley and Travers however went on to say that high school students may be at the level of formal reasoning although incapable of proportional reasoning. Both Renner and Karplus used examples which required the ratio 3:2. Piaget's generalised descriptions of the late formal replies to the proportion tasks in "The Growth of Logical Thinking" involved the realisation that the task required a multiplicative strategy and the ability to handle three variables. In the similarity of rectangles task the child at stage four (which might be early formal since the ages are 11 - 12) is allowed to build up to an answer, one child is quoted using a multiplicative method with the ratio 12:5 however.

Karplus regarded a correct solution to the Mr. Short problem when a concrete comparison of lengths of paper clips and buttons was used as a less sophisticated method than one which was purely quantitative. Renner's next general category involved the recognition of the ratio 3:2 but used incorrectly, this would compare with Karplus' transitional stage in which is also included the concrete referent method just described.

All three researchers mentioned the enlargement by an additive strategy rather than a multiplicative one. Piaget described the use of the data to provide the additive factor as substage IIB; Karplus did not see this strategy as appearing on a continuum leading finally to the use of the correct method. In discussion of the eel question Piaget described enlargement by the addition of one or a constant amount not available from the data. Renner tended to put these attempts into a low and intuitive category as did Karplus. Karplus however singled out the child who realised that multiplication was needed but chose a scale factor (usually 2) which was not present in the data. Both Karplus and Piaget noted that the ability to cope with the ratio 2:1 was not a sign of true proportional reasoning. The most naive categories on all the tasks tended to be either the enlargement of just one dimension disregarding the other, a realisation that the figure must get larger but taking no notice of the proportional aspect and finally (from Piaget) the refusal to measure at all even though this was essential to the problem.

The Significance of the Ratio Used in the Tasks

Mention has already been made of the accessibility of strategies to handle the ratio 2:1. Noelting (1978) reported a study in which the task stayed essentially the same but the ratios involved were varied. The task involved mixtures of water and orange juice, the child had to state the relative taste of the mixture. The sample was composed of children from six to sixteen years of age and totalled 321. Mathematically advanced classes were chosen at each level and each class was from the upper middle class socio-economic group in Quebec City. The items were ranked according to difficulty, a Guttman scalogram analysis was used to test for scalability and then different groups of items were designated to be comparable to those demanding different Piagetian levels of thought. The ratios involved are listed below together with Noelting's mathematical descriptions.

Table 1. Comparison of Different Mixtures (Noelting 1978)
Items of Orange Juice Test (Group Form A)
Ordered According to Degree of Success, then Categorised to Form Stages

Stage	Item	Composition	Frequency of success	Characteristics
0	0	(1,0)vs.(0,1)	-	Differentiation of terms.
IA	2	(4,1)vs.(1,4)	319	Difference between first
	6	(3,1)vs.(2,2)	319	terms of ordered pairs.
	4	(1,2)vs.(2,1)	319	
IB	1	(1,0)vs.(1,1)	311	Like first term, difference
	3	(1,2)vs.(1,5)	307	between second terms of
	5	(1,1)vs.(1,2)	305	ordered pairs.
IC	8	(2,3)vs.(1,1)	295	Equality vs. difference
	13	(2,1)vs.(3,3)	291	between terms of
	10	(2,2)vs.(3,4)	297	ordered pairs.
IIA	9	(2,2)vs.(3,3)	251	
	11	(1,1)vs.(3,3)	244	(1,1) equivalence
	7	(1,1)vs.(2,2)	231	class.
IIB	12	(1,2)vs.(2,4)	186	Any equivalence
	15	(4,2)vs.(2,1)	156	class.
IIIAI	16	(2,1)vs.(4,3)	141	Ordered pairs with
	17	(1,3)vs.(2,5)	131	two corresponding terms
	14	(2,3)vs.(1,2)	107	multiple of one another.
	18	(2,1)vs.(3,2)	88	
IIIA2	20	(6,3)vs.(5,2)	87	Same after simplifying
	22	(4,2)vs.(5,3)	71	one pair or extracting
	19	(2,3)vs.(3,4)	65	(1,1) unit.
	21	(3,2)vs.(4,3)	59	
IIIB	23	(5,2)vs.(7,3)	51	
	24	(3,5)vs.(5,8)	-	Any fraction.
	25	(5,7)vs.(3,5)	-	

Piagetian terms.

Table 2: Comparison of Age Distribution at Each Stage of Orange Juice Test (Group Form A) (Noelting 1978)

Age	N	Stage 0	IA	IB	IC	IIA	IIB	IIIA	IIIB
6	14	0	1	2	8	3	0	0	0
7	26	1	1	7	14	2	1	0	0
8	35	1	0	4	12	10	6	2	0
9	43	0	1	2	9	12	13	6	0
10	32	0	0	1	3	13	8	6	1
11	38	0	0	1	5	12	7	9	4
12	34	0	3	1	0	9	5	14	2
13	31	0	2	0	0	2	9	17	1
14	20	0	0	0	11	1	2	10	6
15	29	0	0	0	0	0	8	16	5
16	19	0	0	0	0	1	2	8	8
Total	321	2	8	18	52	65	61	88	27
p [a]		-	-	-	-	<.01	<.01	<.01	<.01
Age of accession [b]		-	-	-	-	8;1	10;5	12;2	(17;0)

NOTES: [a]Probability level of difference between age distribution of the stage, compared with preceding one, assessed by Kolmogorov-Smirnov Test.

[b]Age of accession to a stage is the age where 50% ofSs solve at least one item of the stage.

It can be seen that the comparison of ratios such as 5:2 and 7:3 was accomplished by very few children, all of them being in the older age groups.

A similar experiment using mixtures was carried out in Israel by Stavy et al (1977). They hypothesised that there were four phases to proportional reasoning. Their experiment used sugar concentration in water, they added a specific amount of sugar (1 or 2 spoons) to a cup or half cup of water, which gave concentration ratios of 2 spoons to 1 cup, 1 spoon to 1 cup etc. The phases they described were: 1) the direct function, an increase in the numerator (sugar) increased the ratio (concentration); 2) the inverse function, an increase in the denominator (water) decreased the ratio; 3) proportional reasoning where both the numerator and the denominator varied; 4) the realisation that the concentration was a non additive physical quantity e.g. if two solutions having the same concentrations were poured into a third container the same concentration ratio was obtained. Their findings appeared to indicate that there were different developmental patterns for each of the four phases. The direct function was solved by practically all the children, the inverse function and proportions seemed to be solved nearly simultaneously, the intensive quantity task was marked by a U shaped curve (success against age). The ability of the youngest children to solve this task (age five) was linked with their global view of the problem when they concentrated not on the elements but on sweetness, then there was a lack of success until about age nine when the correct solution arose from a coordination of reasons.

<u>The Understanding of Ratio in Young Children</u>

Muller (1978) supported to some extent the criticism of Piaget's interpretation of the development of proportional reasoning put forward by Bryant (1974). Bryant showed that young children could make proportional type inferences and stated that the difficulty in making proportional judgements was a result of making incorrect initial analyses rather than a failure to make an overall inference. Muller presented a different task to the children (aged five to eleven years) which he believed was free of the one to one correspondence clues apparent in the Bryant work. Muller concluded that young children are capable of logically connecting.....

> two discrete perceptual experiences by constructing a common identity element, be it size, colour or proportion. The problem for the young child is, as Bryant suggests, that of making the correct initial analysis, in this case choosing between size, colour and proportion and deducing that proportion is the only consistent clue. p.34.

This supported Noelting's experience that some aspects of ratio are understood by young children.

Van den Brink and Streefland (1978) argued that young children used a framework of natural standards. They reported both classroom experiments and discussion with children. The latter involved a discussion on the correctness of a cinema poster showing a killer whale, when the child declared that the whale was too large and called upon his memory of seeing a whale a year before. He compared his memory of the whale with the size of a man and then applied this approximate ratio to the sizes of the whale and man in the picture. They also reported classroom evidence of the same phenomenon. Children were shown a picture of a house and asked to show their own height in comparison to the height of the house - "Where would your head come up to?". The children spontaneously used the door of the classroom as a comparative measure, that was employing child height : door of classroom, door of classroom : door of house and so child height : house height. The picture was a trick however and when projected a second time it was seen that a child was larger than the house, the children in the class immediately described the house as a doll's house.

Teaching Ratio and Proportion

Fischbein and fellow authors (1970) investigated the concept of chance with children who were pre-school or in the third or sixth grade. The problem presented was that of choosing a marble of a particular colour from a mixture of marbles of two colours. Three forms of instruction were given, one merely illustrated the possible responses, the other two were designed to instruct the subjects on a solving procedure. Fischbein reported that the spontaneous responses of nine and ten year olds scarcely differed from that of the pre-school children but after "brief systematic instruction, their responses became comparable with those of 12 to 13 year olds Ss. They became able to estimate chances by comparing ratios correctly". p.387

Karplus having spent some time reporting the incidence of the addition strategy in the answers given by children to the Mr. Short problem, declared that this particular strategy was not part of a continuum of development towards the correct use of proportion. Kurtz and Karplus (1977) reported on a teaching experiment for ninth and tenth grade pre-algebra students. Two teaching programmes were used, one involving the laboratory approach using tangible examples of the constant ratio, the other used a paper and pencil format of the practical tasks. The authors distinguished between constant difference, constant ratio and constant sum, the activities being concerned with constant ratio some sixty percent of the time. The first part of the experiment involved the distinction by the children of each of the three types of relationship, they were given tables of data with omissions and asked to supply the missing information, write an equation and graph the data, commenting on the families of curves produced. The second phase presented ten practical situations, one set in written form the other in practical experience. Part three led up to the presentation of proportional problems in the traditional format of three knowns and one unknown. Items used on the pre and post tests included three dealing with constant ratios and one with a constant difference (One sister is six, the other is twelve, when the younger is nine, how old is the other sister?). The gains of the two experimental groups over the control group between pre and post testing were substantial. In the control group over sixty percent of the subjects responded at the same level at each testing. The experimental groups showed substantial gains to an algebraic procedure on the immediate post test but moved from the algebraic approach to a multiplicative proportional strategy on the delayed post test. At all times the vast majority of students showed the correct additive strategy on the age question. Thus the authors have shown that it is possible by a carefully designed teaching programme to advance the level of proportional thought, although the algebraic representation of proportional reasoning did not appear to be internalised.

Renner (1977) also reported that after a course of science in which the student used exploration and a discovery approach before any verbalising of concepts took place, college students showed a marked progression on the Karplus task. Out of forty-four students tested nineteen scored the maximum on the pretest, thirteen students gained stages on the post test and three regressed.

Abramowitz (1975) stated that the child's understanding of proportionality was dependent on the task being performed. In particular in her doctoral dissertation (1975 a) she distinguished between four dimensions on tasks of the form a/b = c/x. The four were : the size of a/b, equality or inequality of b and c, complex or simple fractions and the form of the test. In another paper she compared performance on proportion problems with performance on questions dealing with fractions (1975 b):-

> A surprising result from the factor analysis is that skill tests of facility with fractions load on a different factor than tasks involving proportionality..... proportion tasks demand a knowledge of this facility but also an understanding of how to and when to use it in an appropriate situation......It suggests that drill alone may be insufficient in teaching proportion. The teaching of fractions must be supplemented with tasks which help students conceptualize what they are doing with these numbers.

Steffe and Parr (1968) tried to partition out the potential cause of difficulty due to the mode of presentation of a proportion problem. They reported that:

> 1. There is little correlation between the ability of children at the fourth, fifth and sixth grades to perform successfully in proportionality situations at a symbolic level, such as $\frac{6}{15} = \frac{\square}{5}$ and their ability to perform successfully on proportionality situations based on ratio or fractional pictorial data.
>
> 2. Children solve many proportionalities presented to them in the form of pictorial data by visual inspection both in the case of ratio and fractional situations.
>
> 3. Whenever the pictorial data, which display the proportionalities, are not conducive to solution by visual inspection, the proportionalities become exceedingly difficult to fourth, fifth and sixth grade children to solve, except for the high ability sixth graders.
>
> 4. For the denominator test, the proportionalities represented pictorially by a ratio situation were easier for the children to solve than the proportionalities represented pictorially by a fractional situation.

> 5. The children of high intelligence are much more adept at solving proportionalities for both a symbolic and pictoral representation than are children of low intelligence.
>
> 6. The fifth and sixth graders performed significantly better than the fourth graders on all tests and subtests involved. p.26

As implications, Steffe and Parr noted:

> 1. Much more care must be taken in the fifth and sixth grades to develop a sequence of lessons which are designed to enhance children's ability to represent visual data mathematically in the case of ratio or fractions, indeed if that ability can be enhanced.

Vergnaud and the workers of IREM, France (1977) have investigated what they call "Isomorphisms between measures, otherwise called problems in proportions", they outlined two procedures available when the problem presented was of the form:

$$\begin{array}{c|c} a & b \\ c & x \end{array}$$

The scalar procedure allowed one to pass from b to x in the same way as passing from a to c.

Scalar

$$\begin{array}{c|c} a & b \\ c & x \end{array}$$

Functional

$$\begin{array}{c|c} a & b \\ c & x \end{array}$$

The functional procedure allowed one to pass from c to x in the same way as one passed from a to b. By eliminating the factors of size of number involved and the underlying physical concepts the experimenters attempted to see which of the two procedures was more readily used by the children. One hundred and twelve children aged twelve to sixteen were tested. The scalar procedure was used more than the functional procedure. No inferences could be drawn as to the result being a spontaneous choice or an out-come of the methods of teaching used with the children.

<u>Summary</u>

The research literature shows that although the idea of ratio and proportion is understood by young children when the ratio is 2:1 or a repeated application of 2:1, the application of more complex ratios is very difficult for them. The methods the children use to solve such problems are usually based on an additive strategy, sometimes correct and sometimes in the sense of the Karplus additive strategy, plausible but incorrect. There is a clear distinction between the ability to cope with 2:1 and 5:3 for example. Freudenthal (1972) in fact suggested that:

> The pupil should learn to tackle such problems by intuitive means, even 'ad hoc' means, and he should do it in a diversified way. With respect to these concepts such intuitive methods are again bottom level pre-mathematics..... One should not banish them from elementary arithmetic but one should not cultivate them there either as was often done in the past. p.276

Freudenthal proposed the use of algebra as the solution to the difficulties apparent in the concept of proportion.

Whether one considers the use of proportion to be a formal level task, in the sequence of levels postulated by Piaget or as an operation requiring a second order relation (being a relation between two relations), it is certainly difficult for children and shows on most studies a clear correlation of measure of success with age. There are studies to suggest (Karplus, Renner) that by showing the child the application of ratio and indeed involving him in the experiments, his performance will improve.

Research Related to Hierarchies of Mathematical Understanding

Background

When education moved from the viewpoint that children were simply small adults and that their education should be entirely tailored to their needs as an adult, to a more child-centred approach in which childhood was seen as a state in itself with special needs, then educators started to consider the appropriateness of material for children of different ages. In recent years this problem has been approached in two different ways both based on the idea that there exists some form of hierarchy of learning. The cognitive approach affirms that there are behaviours peculiar to a particular type of understanding, either in that the child displays them spontaneously for a while and then moves on to a different type of behaviour (Piaget) or that the child finds it more acceptable to react to certain forms of presentation at certain stages of development (Bruner, Bloom's Taxonomy). The skill orientated approach of Gagné on the other hand takes a task which is to be taught and breaks it down into small sections each of which is considered a prerequisite:

> Each learning set in the hierarchy is represented by a distinct class of tasks, and measured in the individual by one or more representative tasks from this class. In order for learning to occur at any point in the hierarchy, according to this theory, each of the learning sets subordinate to a given task must be highly recallable, and integrated by a thinking process into the solution of the problem posed by the task. The attainment of the final task is thus conceived to be a matter of successive attainment and 'integration' of a series of lower level learning sets, beginning with those which are already available to the individual. (Gagne and Paradise, 1961, p.2).

The latter part of this statement is very similar to that made by the Piagetian school on assimilation and accomodation; in practice the difference between the two approaches is that in the cognitive approach one tries to find out where the child is, using ingenious and non-taught tasks whereas in the skill approach one wants to move a child to a predetermined goal and one decides how to go about this.

With the advent of teaching machines and programmed learning the demand for a sequence of instruction was increased (a sequence of instruction meaning the order in which the learner interacts with the units of content). The problem appeared to be very much more complex however than simply lining up skills. Suppes (1966) said:

> My present view, based partly on our experiments and partly on conjecture, is that the psychological stratification of math. concepts will seldom if ever, do violence to the logical structure of these concepts; but it will markedly deviate from the mathematical analysis of the same concepts with respect to the amount of detail that must be considered. p.145.

Helmer (1969) reported on the attempts to validate a sequencing of instruction in various mathematical topics, these presented the sequence in the postulated order and then in a scrambled version to a comparable group of students. Roe, Case and Roe (1962) reported no significant difference between the two groups on an immediate post test. Roe (1962) repeated the experiment using an extended form of the instructional sequence; he found that the group using the scrambled version performed significantly worse. The size of the unit being scrambled and the amount of redundant information in the sequence may be crucial aspects of the sequencing.

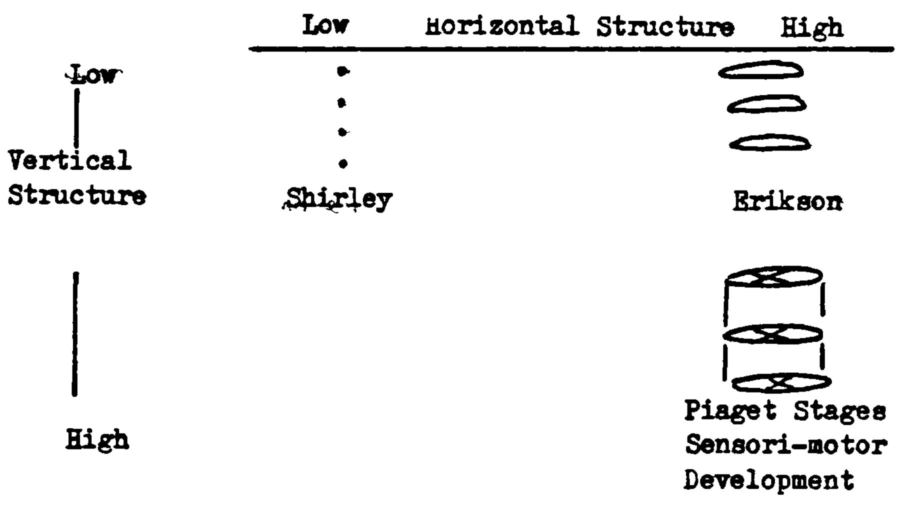
The cognitive approach is very much broader since it takes the children as they are with whatever experiences they have already had and tries to describe stages of development which can be applied over a wide range of activities. Wohlwill (1973) described "stage as a construct within a structurally defined system, having the property of unifying a set of behaviour." :-

> The underlying assumption is that in certain areas of development, particularly in the cognitive realm, but not necessarily confined to it, there exist regulating mechanisms that modulate the course of the individual's development so as to ensure a degree of harmony and integration in his functioning over a variety of related behavioral dimensions. The mechanism might be thought of in part as a mediational generalization process, permitting acquisitions in one area, for example number conservation, to spread both to equivalent aspects of different concepts (e.g. conservation of length) and to different aspects of the same concept (e.g. cardinal-ordinal correspondence). The result is the formation of a broad structural network of interrelated concepts appearing, not all at once to be sure, but within a fairly narrowly delimited period, with further progress along any component concept or dimension being assumed to be deferred till the consolidation of this network - that is, the attainment of the "stage". Wohlwill (1973) p.192.

Wohwill described the researches of Shirley and Erikson and compared them with the idea of stage as described by Piaget. In Shirley's

Motor sequence (1931) the researcher described a sequence of behaviours characteristic of the development of locomotion in infancy. They formed a set of highly concrete, specific motor response patterns. A series of qualitatively differentiated modes of motoric activity appeared in a predictable order during the course of the development of the infant. In Erikson's phases of development (1959) each phase referred to a constellation of emotions, feelings and dispositions which did not have any direct reference to overt behaviour patterns. Each phase was useful in describing commonly encountered sources and types of emotional conflict in the growing individual's personal and interpersonal life, and in ordering these along the various segments of the life cycle. They each fell short of the criteria for the concept of a stage i.e. sequentiality, hierarchical integration of lower into higher levels, gradual consolidation of stages in formation, structural whole character (unifying interrelated behaviours, concepts or skills) and equilibration. These being the requirements stated by Piaget. Wohlwill illustrated the difference between the three sequences of behaviours with the following diagram:

Figure 2. Wohwill's Illustration of Three Sequences of Development.



Considerable research has taken place purporting to validate the developmental sequence of stages as described by Piaget (see Pinard and Laurendeau's discussion of stages, 1969). This review of the literature describes only those Piagetian researches which use some form of statistical analysis to test a sequence of mathematical learning, since they more closely parallel the present study.

Developmental Sequences in Learning Mathematics

Developmental sequences in the understanding of mathematics which are not simply validating a Piagetian hierarchy have been suggested by Dienes (1972) and Collis (1975). Lunzer described the Dienes suggestion thus:

> On a still more general plane, Dienes (1972) describes what he sees as six steps in the process of learning mathematics. The first is a stage of free play in which the learner is confronted with a suitably structured mathematical environment and is free to discover what can be done with the material. The second consists in the introduction of constraints or rules. In effect these will usually correspond to manipulative sequences which correspond to practical equivalences. The third stage introduces a variety of isomorphic materials designed to facilitate abstraction of mathematical structures. A fourth stage provides a kind of mathematical image for the abstraction in the form of graphs, Venn diagrams, etc. The fifth stage introduces a symbolism which will enable the learner to represent sequences of transformations. The sixth and final stage consists in devising a suitable set of axioms and theorems to provide a complete description of the system under consideration (quoted from Lunzer, 1973, p.15.)

Collis (1975) expanded the stages put forward by Piaget, describing them in a mathematical context and adding distinctions (within the context of generalised arithmetic) which took into account the child's view of the role of numbers. The four stages of development stated were:

1) Early concrete operational (age 7-9). The elements and operations in ordinary arithmetic are related directly to physically available elements and operations. There are no more than two elements and one operation in any problem. The only inverse available is physical e.g. what is put down can be taken up, in a subtraction situation.

2) Middle concrete operational (age 10-12). The children can work with operations but only where the uniqueness of the result is guaranteed. More numbers with the same operation or larger numbers with the same operation can be dealt with. The notion

of inverse is more developed in that subtraction is seen as the result of destroying the effect of addition but the result is unique.

3) Late concrete operation (age 13-15). The children can work with operations and abstractions but the uniqueness of the outcome must be guaranteed. The inverse is seen as an undoing of an operation. The children are not necessarily happy to use letters.

4) Formal operation (age 16+). The children come to the problem with a set of abstract hypotheses to test, they can manipulate abstract variables which may have a bearing on the solution to the problem. They regard the inverse process as looking directly at the operations themselves without necessarily affecting earlier operations.

The child's view of the outcomes of operations hinges on the idea of "allowing lack of closure". At the first level (age seven) two elements connected by an operation must be actually replaced by a third element. Small numbers must be used with a single operation and the whole must be related to the physical world. At about the age of ten the child recognises that the outcome of the operation is unique, it is not necessary to make a replacement but it is recognised that there is one. The child can use numbers outside his verifiable range. At twelve the child can refrain from actual closure as long as he feels that a unique familiar result is available when required. He is capable of working with a formula such as V=LxBxH as long as he considers each letter stands for a unique number and each binary operation can be closed at any time. At the final stage (age fifteen) the child can work with the operations themselves and does not need to relate the operations or elements to reality. He can hold back closure until various possibilities have been tried, for example in V=LxBxH he can discuss what happens if L is increased and B decreased

As Lunzer (1973) said of the dissertation presented by Collis:

> Collis incorporates two dimensions within a single model of development. One of these relates to the form of a

> problem, the other to its content. When the content is both specific and intuitable it is classed as 'concrete'. In the context of mathematics this reduces to a condition that the problem must be confined to the manipulation of small integers. When the numbers are large or when they are replaced by algebraic variables, the content is 'formal'. As to the operational structure of a problem, this is concrete when it takes the form of a series of closed operations i.e. even if the solver is required to execute more than one operation, he is always able to complete the one he is executing before proceeding to the next, the sequence is clear from the outset. When these conditions do not obtain the structure is formal. This twofold classification leads to a two by two matrix. 8x3 = 3x? and 8x4 - 4 = ? are examples of concrete form and concrete content. axb = bx? and 428+517 = 517+? are examples of formal content in a concrete operational structure. 7-4 = ?-7 is an instance of concrete content in the context of a formal structure. Finally 576+495 = (576+382)+(495-?) and a/b = 2a/? are (relatively simple) instances of formal content in a formal structure. It should of course be stressed that the problems can only have a formal structure for the solver just so long as he has not been taught an algorithm for their solution. If he has, then he knows just what sequence of operations he should perform, and so, by definition, the problem ceases to be formal. p.13.

Research on Sequence of Task Levels (from Piagetian Sources)

In this study the review of research on sequences is restricted to the tasks of Piaget and particularly concentrates on those studies which use a statistical technique to ascertain the existence of a hierarchy. This restriction is in order to outline the information available on the methods for testing hierarchies (particularly when the need is that of ordering child performance). In research of this type one starts with a set of tasks and a hypothesis (based on theory) of the order in which they will be successfully completed.

Kofsky (1966) investigated a sequence of classificatory skills with children at each age level from four to nine years. The children were above average intelligence. The study was carried out by interviewing the children for between ½ and ¾ of an hour (the younger children having two 20 minute sessions). The interviewer used a set of coloured blocks. The Piagetian hypothesis is that the order of successful completion of classification tasks would be as follows:-

1/A Resemblance sorting (matching on the basis of form, colour or some perceptual property).

2/B Consistent sorting (selecting 3 or more objects which are alike in some perceptual feature).

3/C Exhaustive sorting (child consistently uses an attribute to select and selects all blocks with that attribute).

4/D Conservation (blocks are invariant in their class no matter what changes in position etc. are made).

5/E Multiple class membership (Can objects belong in more than one category?).

6/F Horizontal classification (sorting twice (two different criteria) so that all members of a group share the attribute and all possessing the attribute are in the group).

7/G Hierarchical classification (child must classify all members of group by a shared attribute and a subset of the group by another attribute).

8/H Some and all.

9/I Whole is the sum of its parts. (Would we have the same size tower if I used all the reds and all the blues and you used all the squares? (only red and blue squares available)).

10/J Conservation of a Hierarchy. (as above but if I took away all the reds, are there just blues left, just squares or both blues and squares).

11/K Inclusion.

Having tested the children and obtained the frequency of success Kofsky firstly split the tasks according to facility levels; this provided six groups. She then compared the order of facility with the hypothesised order, the correlation was significant at the $p < .01$ level.

The second requirement was to test whether the mastery of a particular task necessarily showed that all logically previous tasks had been mastered. By looking at pairs of items and computing a Loevinger Homogeneity Coefficient which uses the cells pass/fail in the four cell matrix

Fig 3. The Four Cell Pass Fail Matrix

		Item 1 pass	Item 1 fail
Item 2	Pass		
	Fail		

It was seen that only a few response patterns (a third) showed success at harder meant success at easier. The other items were not well-ordered. The pattern of success for each child was then investigated. There should have been twelve patterns if success at the hardest pre-supposed success (in order) at the less difficult. In fact only 27 percent of the sample showed perfect patterns, there were 63 patterns in all. Ms Kofsky repeated the procedure, eliminating items, but still only obtained perfect patterns in 51 percent of the cases. She concluded that the experimental conditions of the tasks may affect children in different ways and that a child may not fit exactly on a continuum but be capable of a wide range of behaviours.

The ordering of tasks by facility and the comparison of the success rate of the younger and older children provided some validation of the theory of Piaget. When however one looked at the consistency in the performance of an indiviual child the theory was not as tenable.

Nassefat (1973) selected 150 children in the age range which should show a transition from concrete operations to formal operations in Piaget's terms. He administered a set of 48 items dealing with both concrete and formal operations. Each item was classified according to which of these two operations was required and one-third of the items were assigned to an "Intermediate" category. Response categories were combined after an initial marking, to form a four-point scale. This reflected pass or failure combined with the abstraction by the child of relevant from irrelevant data. Association between response type and age level was assessed by using Kendall's tau; the values of tau were mostly highly significant. Nassefat then investigated the scalability of the data (in the Guttman sense) based on pass/fail responses. Scalability was assessed separately for each age level. The coefficients used were Loevinger's index of homogeneity and Green's index of consistency. Green's index is similar to both Loevinger's coefficient and Menzel's coefficient of scalability. Consistency (C, I, F category/age range) was generally highest on concrete items at age nine, intermediate at age eleven and formal at age twelve; although in the formal category the consistency never exceeded .25. It is argued that this reflected the fact that only a minority of the oldest subjects pass them. Nassefat regarded the homogeneity of the items in a stage as a sign of the stability of that stage in a child.

A further analysis of the types of response was carried out; this showed the most advanced type of response increased with the age of the child and was highest for Concrete items and lowest for the Formal type. Responses which showed a correct abstraction of the essential information but faulty deduction, occurred at a maximum at precisely those ages at which homogeneity was a minimum. A contingency table was set up for each age group to show the association between responses for any two items. Nassefat chose to assess association by using Kendall's tau, rank ordering the subjects in terms of their responses to each item, tau being calculated for these pairs of rankings. There appeared to be considerable consistency of response for the eleven and twelve year olds on intermediate items and a lowering of consistency during what might be considered a transitional period (age ten for intermediate and age eleven for formal items). In a later article Wohlwill (1973) further criticised the use of tau and the acceptance of a .10 significance level.

Wohlwill (1960) himself investigated the development of the number concept in young children (age four to seven years). The method used was that of number matching; the child was asked to match a number to a displayed one. An initial training period occurred in which the child had to successfully match the numbers 2, 3, 4 shown as dots on cards to other configurations of the same numbers. For the actual experiment there were seven tests each varying in Piagetian definition. The sample size was seventy-two and the number who passed each test was as follows:

Table 3: Results of Wohlwill's Study (1960)

	No. passing	N = 72
B. Elimination of perceptual clues	49	
A. Abstraction	46	
C. Memory	32	
F. Addition and Subtraction	24	
D. Extension	21	
E. Conservation	14	
G. Ordinal-Cardinal Correspondence	6	

(The postulated order was A,B,C,D,E,F,G)

A rank order correlation between predicted and observed order was .86. A pass on a set of trials was taken as 5/6 or 10/12. Subjects who attained the 5/6 were given three additional trials but since they almost invariably succeeded on these items, the tougher pass mark of 8/9 was abandoned in favour of 5/6.

Wohlwill now investigated the scalability of the tests. He used both Loevinger's H_t and Green's Index of Consistency (I), on this data they had very nearly the same value, $H_t = .620$, $I = .616$. He took an index value of above .5 as an indication that the tests represented a single scalable dimension. The deviation from a perfect scale ($I = 1.00$) was considerable however. Only forty-five of the seventy-two had perfect scale type responses, that is their response pattern could be ascertained from their score. When those who either had everything correct or everything incorrect were deleted, only half of the remainder had perfect scale type responses. Wohlwill then looked at each test in turn and found the Loevinger Coefficient H_t which tests the homogeneity of each test with the total set of tests, showing the power of the individual test to discriminate subjects with a relatively higher total score from those with a relatively lower total score. A, B, D, E, F had $H_t \geqslant .75$, for C the H_t was .524 and for G, $H_t = .48$.

Wohlwill concluded that the data did not warrant the acceptance of each test representing a discrete, well determined point on the scale of conceptual development. Lack of high H_t values for each level and the fact that there was no significant separation between various pairs of adjacent tests, were given as the reasons. He then put forward reasons (concerning the actual test) for rejecting one test altogether and he then grouped two tests (A and B) together. The latter strategy showed that the 29 subjects who failed A or B or both accounted for only five passes on all of tests D, E, F, G.

Schwartz and Kofsky-Scholnick (1970) investigated the Conservation of Discontinuous quantity (pouring sweets from one glass to another). They used Elkind's (1967) description of the judgements required to successfully solve a conservation task. The Piaget Conservation task requiring a verbal response to "who has the most" was used as a pretest and post-test after both training and experimental tasks requiring no verbal response. The experimental tasks involved (a) two wide identical glasses and two narrow identical glasses with firstly equal amounts of sweets and then sweets in 2:1 ratio (b) one wide and one narrow glass filled to same height (c) glasses with both diameter of glass and level varied (d) emptying one glass into an identical one (e) pouring from narrow glass into a wider one (f) two identical glasses filled to same level, then one poured into an identical glass (g) pouring from a narrow glass into a wide glass having at the start two filled identical glasses. A three-way analysis of variance was carried out to test whether there was significant interaction between the stimulus and subject and stimulus and type of judgement used. There was such interaction. All seven tasks were subjected to a test of scalability using H_t (Loevinger) and I (Green), the pass mark for any task (sometimes 4 parts) was 100 percent correct. Both the coefficients H_t and I had values over .9 and only two children out of the 40 in the experiment had erroneous response patterns.

The following researches concerned with the formation of Hierarchies do not refer to mathematics but use statistical techniques for testing scalability. Peel (1959) reported three experiments in which tasks were stated as requiring certain Piagetian levels of thought. The data obtained from giving children these tasks were subjected to scalogram analysis. One experiment (carried out by Miss Tern) repeated tasks of Piaget concerning the child's treatment of elementary spatial relationships in drawing. She used 55 children aged 2 years, 9 months to 7 years, 9 months. The order of 21 tasks was hypothesised according to age level. This produced some reversals e.g. John was older than Clive but appeared to score less. The 21 tasks were then formed into four groups suggesting Piagetian stages I, IIA, IIB and III. When a Güttman scale was used, 28.3 percent error response patterns emerged, with a Coefficient of Reproducibility of .717; Peel suggested that a Coefficient of Reproducibility of .75 was adequate, Guttman's being too stringent. This would allow one to accept 25 percent error responses. Peel suggested the Coefficient of .717 lent support to Piaget's ordering of tasks. When Miss Tern ordered the children according to total score, the reproducibility coefficient was .835. The suggestion was that drawing of individual items correlated highly with the total overall drawing level of the child.

The other two experiments used the same type of analysis, tending to keep the task order invariant but varying the criteria for the ordering of child performance i.e. (a) age level, (b) mental age level, (c) total score.

Goldman (1964) carried out a very similar type of experiment on children's responses to a bible story (age 6 years, 1 month to 17 years, 11 months). Levels of response corresponding to Piagetian stages were not available so he submitted answers on one question to a group of five judges who were asked to grade each response according to how it would fit into a pre-operational/formal scale. Disagreement by one stage between the experimenter and a judge was resolved by taking the higher level, a two level disagreement was resolved by taking the average. Goldman combined categories to obtain three levels and again subjected the data to a Guttman Scalogram Analysis taking .75 Reproducibility as adequate. The children were ordered again on age, mental age and total scores, the latter giving the highest coefficient. Different coefficients resulted from different biblical stories.

Thus we see that where a hypothetical order of Piagetian tasks already existed, the scaling attempts could be made using facility, age or total score. A low coefficient of reproducibility might result in either re-ordering the tasks or in changing the criteria for scaling from age to total score etc. In some of the research, although individual tasks matched the hypothesised order of difficulty, it was found that a child often succeeded on a harder item without succeeding on all easier ones. If one regards the requirements for the existence of a 'stage' in a developmental sequence to include scalability then simply ordering according to facility is insufficient.

CHAPTER THREE

Construction of the Test

Purpose of the Study

The CSMS Project, funded by the Social Science Research Council was set up in order to investigate a hierarchy of understanding in secondary school mathematics. In order to do this the mathematics team decided to test secondary school children on a number of different mathematical topics, Ratio was one such topic. In order to test a large sample of children the testing had to be carried out using a paper and pencil format. The construction of the test instrument is the subject of this chapter. The study on Ratio and Proportion was carried out for the following reasons:

1. To formulate a series of problems which mirrored the topic of ratio and proportion as it is seen in the secondary school curriculum and amplified other research .
2. To interview a number of children of secondary school age to see which methods (both correct and incorrect) they used to solve ratio and proportion problems.
3. To test a large representative sample of school children of different ages in the secondary school to find the levels of attainment on the problems mentioned above as well as the incidence of specific errors.
4. To group the items on the ratio paper according to facility and homogeneity, thus giving evidence for descriptions of different levels of difficulty within the topic of ratio.
5. To compare the ratio levels with the levels of difficulty within other mathematical topics and finally with cognitive demand on a Piagetian scale.
6. To verify the grouping of the items based on the 1976 data by a further testing in 1977 and from the results of a two year longitudinal survey.

The results should be of use to teachers and developers of curriculum in that they demonstrate the difference in demand of various parts of the topic of ratio as well as showing the percentage of children at each age level who have attained each level of understanding. The knowledge of the incidence of specific errors should alert teachers when teaching the topic. The insight gained from the interviews should help teachers understand the methods children use to solve problems in ratio, in that they can build on those methods rather than assume that the child is without a method unless using the teacher-taught one.

The study consists of several phases; firstly before any attempt was made to test children an analysis was made of the topic of ratio and proportion as it appears in British secondary school textbooks and how it was seen by other researchers. The final outcome of the study was to be wide scale testing so an important phase was the writing of the test instrument, information on the suitability of the items was obtained from interviewing children. The phase during which children were interviewed therefore served two purposes, one to inform so that modifications to the test items could be made and two to ascertain the methods the children themselves used to solve the problems dealing with ratio. Once a satisfactory form of the items was reached, a written test was given to two hundred London children, this being a pilot study of the suitability of the test.

The first wide scale testing took place in the summer of 1976, a second testing was carried out in the summer of 1977 and the children forming the sample for the longitudinal study were tested in 1976, 77 and 78. A major purpose of the study was the formation of a hierarchy of understanding in the topic of ratio and proportion therefore a method of analysis of the data had to be found.

These phases of the study which occurred prior to the testing of a large sample of secondary school children are reported in this chapter and the next. Both the testing of a large sample and the interviews with children depended on the test instrument. In this chapter the construction and modification of the items is discussed, the modifications being made principally on the basis of the difficulties met by the children when interviewed and asked to work out the problems. The results of the pilot testing (with a sample of two hundred children) are also reported in this chapter.

The Analysis of Textbooks

The topic of Ratio and Proportion as it appears in the British Secondary School Curriculum

Information was to be given to teachers, if the results were to be of practical value it seemed advisable that the interpretation of ratio and proportion should be recognisable to the teacher i.e. not unlike that which was currently taught. An analysis was therefore made of those aspects of the topic which appeared in commonly used secondary school textbooks.

Fractions form a large part of the mathematics curriculum in the early years of the secondary school, such a large part in fact that the topic of Fractions was regarded as worthy of a separate investigation, therefore the manipulation of rational numbers per se did not appear on the ratio test. Percentages did form part of the test together with aspects of enlargement, similarity and the use of proportion but a decision was made to exclude Trigonometry since this is introduced quite late in most textbook series.

A search was made of some of the textbooks used most frequently in British secondary schools, the series chosen were: School Mathematics Project Number Series 1, 2, 3, 4; School Mathematics Project Letter Series A-H; the books of the Scottish Mathematics Group 1-5; "Pattern and Power of Mathematics" (Moakes) Books 1-7; Midland Mathematics Experiment, GCE and CSE and "New Mathematics" (Knight) books 1-4. In addition various Science books particularly Nuffield (Biology, Chemistry and Physics) and the project "Science Uses Mathematics", were investigated in order to ascertain where ratio and proportion appeared in the science curriculum. An analysis of the topic of ratio and proportion as it occurs in both mathematics and science books appears in Table 4.

Table 4. An Analysis of the Topic of Ratio and Proportion as it appears in Secondary Mathematics and Science Textbooks.

Books:-	SMP A-H	SMP 1-4	MME CSE	MME GCE	Knight	Moakes	Scottish	S.U.M.	Nuffield Biol.	Nuffield Chem.	Nuffield Phys.
Ratio Words and Conventions	D(90) H(55)		1A(311) 2A(81)	1A(311)						Bk 1	Bk 1
Equivalent fractions (lowest form)	D(90) B(51)	2(206)	1A(110)	1A(110)	1(32)	3(108)	1(165)				
Fraction multiplier	H(96)	2(208)				6	3(171)	1			
Inverse multiplier		2(208)				6	3(171)				
Same units	D(90) H(96)	2(206)				3(108)	1(175)				
Written a:b or $^a/_b$	D(90)	2(208)					1(175)				
x:y, y:z find x:z		2(212)									
Scale model → real	E(26)	2(46)	2B(48)		1(149)	2(156)	1(175)	Bio.1-3	1		
real → model	F(82)				1(149)	2(156)	3(179)				
representative fraction					1(149)	5(145)	3(179)				
Gradient	H(6,14) Z		2B(146)	2A(158)	2(44) 4(66)	7	4(115)	Bio.1-3		3/4	
Percentage	D(130)	2(220)	2A(155)	2A(73)		4(99)	1(177)				
Rate Cost	H(10) D(140)						3(171)				2
Speed	Z C(187)			2A(158)	4(59)		3(182)				
Interest					4(195)	4(99)	4(203)				
Conversion			1A(311)	1A(311)	1(149)	6	2(145) 3(171)				
"Rate"								3	2	3/4	1

Table 4. Cont'd.

Books:-	SMP A-H	SMP 1-4	MME CSE	MME GCE	Knight	Moakes	Scottish	S.U.M.	Nuffield Biol.	Nuffield Chem.	Nuffield Phys.
Similar Figures											
Enlargement	D(24)	2(43) 2(51)	1A(183)	1A(183) 2A(36)	1(137) 2(137)	3(101) 3(124) 6	4(101)	4 Bio.1-3			2
Scale factor	D(24) G(85)	2(206)				5(143)		4			2
Area sim.figures	F(82)	2(56)		2A(36)	2(77)	5(143)	4(101)				
Volume sim.figures	F(82)	2(56)			2(77)	5(143)	4(101)				
Recognition sim.fig.											
2D						3(126)					
3D						3(126)					
Triangles	D(24) H(196)			2B(119)	4(75)	3(126)	4(101)				2
Properties of sim. triangles	H(96)				4(75)	3(126)	4(101)				
$\frac{a}{b} = \frac{p}{q} \Rightarrow \frac{a}{p} = \frac{b}{q}$		2(212)				6(158)					
Sharing in ratio	D(90) H(96)	2(127)									
Total number of parts	H(96)					3					
Pie chart	D(100)										

Table 4. Cont'd.

Books:-	SMP A-H	SMP 1-4	MME CSE	MME GCE	Knight	Moakes	Scottish	S.U.M.	Nuffield Biol.	Nuffield Chem.	Nuffield Phys.
Notation α											
Variation					3(101)		5(19)				
Inverse					3(101)		5(9)				
					3(111)		4(4) 5(19)				
'Proportion'		2(212)					3(171)			3/4	1
Defined y = kx	H(55)	2(206)			3(101)		5(19)				
Proportion relation between 2 ratios		2(206)					5A(1)			3	
Unitary Method							3(171-9)				
Direct method					3(101)		5A(1)				
$\frac{x}{y} = \frac{a}{b}$		4				3	5				

Note: All the textbook series start from book 1 or A at the beginning of the first year in Secondary School. The number of the book appears before the bracket and the page number appears within the bracket.

S.M.P. Book A introduces the idea of rate from a situation where something occurs three times per minute, the child is asked to record this as $x \rightarrow 3x$. Book D provides examples of the ratio 2:1 and also the writing in simplest form of 2 to 10 etc. Book F gives statements on the properties of similar figures. Scale drawing appears in Book E. If one assumes that two books are used each year then the introduction to ratio occurs in the second year of the secondary school, no algorithm appears to be introduced. Similarly in the numbered series of S.M.P., book 2 discusses similarity and enlargement, the comparison of fractions representing proportions is used. Similar figures also occur in Book 2. Book 4 produces a method for solving problems involving ratio and proportion, not unlike the algorithm $\frac{x}{a} = \frac{y}{b}$.

The Scottish Mathematics Group series tends to introduce ratio earlier i.e. in Book 1. It also uses formal notation. Similar triangles are not dealt with until Book 4 however. A thorough discussion of direct and inverse variation takes place in Book 5; the finding of an unknown x in the form of $\frac{x}{a} = \frac{y}{b}$ is work set in Book 5. Moakes spends some time in Book 2 on reducing fractions obtained from ratios, to their lowest terms, scale factors do not appear until Book 3.

Knight and S.M.P. use set notation at some stage to describe ratio. There is no strict adherence in any series to an algorithm such as "Unitary Method" or $\frac{x}{a} = \frac{y}{b}$ (three amounts known, one to be found). The examples are varied and involve rate in scientific settings. The requirements of the Science Subjects very often demand a concept prior to its appearance in any of the mathematics texts. The Science subjects themselves do not require the concepts at the same time. Physics always seem to require a particular aspect of ratio earlier than the other science subjects.

The Construction of the Test.

The analysis of the textbooks showed that some aspect of the topic were specific to the text used, it was decided to omit such special cases when an attempt was made to formulate items for use with children. The topics of fractions and trigonometry were also omitted, the former being used in a separate test and the latter being considered both strongly allied to what had been taught as well as of a degree of difficulty unsuitable for the age range being considered. The work of other researchers was considered with a view to using some of the examples they had employed. A start was made on writing items which embodied some at least of the aspects of ratio taught in secondary schools, with a view to producing a pencil and paper test suitable for use in a large survey of children aged 13-16.

The assessment of understanding rather than the testing of known facts seemed to be best embodied in the problem solving format. The problems had to be of varying degrees of complexity in order to provide a challenge for a wide range of abilities, be devoid of as many extraneous difficulties (especially reading difficulties) as possible and yet be sufficiently brief. In order to avoid dependence on what had been specifically taught the language had to be as non-technical as possible thus enabling all children to make an attempt at the questions irrespective of whether they had attended lessons in ratio and proportion or not.

Drafts of the problems which appeared to embody aspects of the topic of ratio taught in the secondary school were discussed by the team, rewritten and in the light of the child interviews written again. Some problems were easily solved by all the children interviewed and so seemed too easy to form part of a short test which was to discriminate between different levels of understanding. Other questions proved to be difficult not because of the inherent mathematics in the problem but because of the form of words or figures involved. After several drafts a final version of the test was arrived at (reasons for rejection of items are discussed below). This test was tried on two hundred children in London schools and the results were analysed and the test further modified.

In the following discussion the items which were later to form the pencil and paper test are presented in their original form and reasons for modification are given. The interviews with children are described only in the light of their relevance to the modification of questions. The complete set of items used in the interviews appears in the Appendix.

<u>The Form of the Interviews</u>

The use of a printed test restricts the amount of information one can obtain about the extent of understanding of each child, it was thought therefore that considerable benefit could be gained by interviewing a number of children using the test as the interview instrument and therefore discovering the methods some children used to solve the problems and also how they failed to solve the problems. Children of different abilities were therefore interviewed on a one to one basis and their replies were used to not only guide the improvement of the test writing but also to provide information on the type of errors one might find in the large sample of children when tested. Thirty five interviews were carried out using a tape recorder and the tape later transcribed. The child was essentially asked to talk his way through a problem and at each step he was asked to explain what he was doing. The children interviewed were from a number of schools and from different ability and age ranges.

Each interview took place in a quiet room made available by the school, the children were chosen by their teachers according to a description of the type of child wanted, given by the interviewer e.g. a third year child who is good at mathematics. Each interview took about an hour although not all children managed to solve the entire set of problems in that time, it was considered that in-depth discussion of a subset of problems was more informative than a hurried account of all of them. Each child was told that he was to be asked some questions on how he did mathematics problems, he was further told that this was part of a research programme which was designed to help teachers in their teaching. He was encouraged to ask any questions he liked and to point out words he did not understand since we were going to send out a test paper and we did not want it to contain words that were not understood.

Usually the interviewer started with what had been found to be an easy item and the child was asked to read it out loud, note was taken of words that appeared to be difficult. Sometimes the meaning of a word was questioned by the interviewer e.g. "Can you tell me what 'the same shape but bigger' means?." Then the interviewer provided two shapes and asked if they fitted the description and why they did. Then the child was asked to do the example and explain what he was doing. The explanation was sometimes prodded by the interviewer with interjections such as "You have written 2, would 3 be just as good?". The child was asked to defend an answer and although sometimes they were wooed away from their original answer by the interviewer's interjections, they were always asked finally to choose which one they preferred and why.

Writing Test Items for the Ratio Test

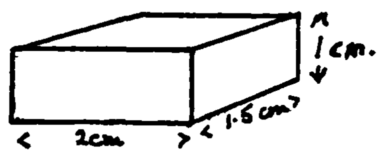
The C.S.M.S. tests were designed to be given by the classroom teacher in the normal mathematics lesson therefore their duration was limited to one class period of about forty-five minutes. The children to be tested were second, third and fourth year secondary school pupils. Various restrictions had to be placed on the number of problems on the paper, the words that could be used and the areas that could be covered in the time available. In all there were five versions of the problem paper before a final version used on the large sample of children was arrived at. Initial planning suggested that there should be the following broad categories of problems:

1. Geometric use of ratio in enlargement, scale drawing and similarity.
2. The use of a fixed rate which in one case was given and in another case had to be found.
3. Percentages.
4. Problems requiring doubling, halving, multiplication by an integer and multiplication by a fraction.
5. The recognition that a ratio had to be used, therefore it had to be found and then used.
6. The recognition of problems where ratio was not needed.
7. Items used in research by other researchers.

Some of these categories were later discarded and the items originally written to fit within a category were also often discarded after children had been interviewed. An important reason for discarding an item was that it either contained matter which required other knowledge e.g. the ability to understand volume was necessary before the actual use of ratio could begin, or that it contained technical terms that the child did not understand; in short where the essential mathematics was hidden by the setting of the question. Certain taught skills such as using a centre of enlargement and a scale factor or the use of an algorithm such as $\frac{x}{a} = \frac{y}{b}$ were not tested but could be used if the child recognised the situation in which they would be advantageous. The following discussion of items is based on the seven categories above, it includes the items which were finally rejected, illustrated by excerpts from child interviews showing some of the reasons for rejection.

Geometry - Scale Drawing

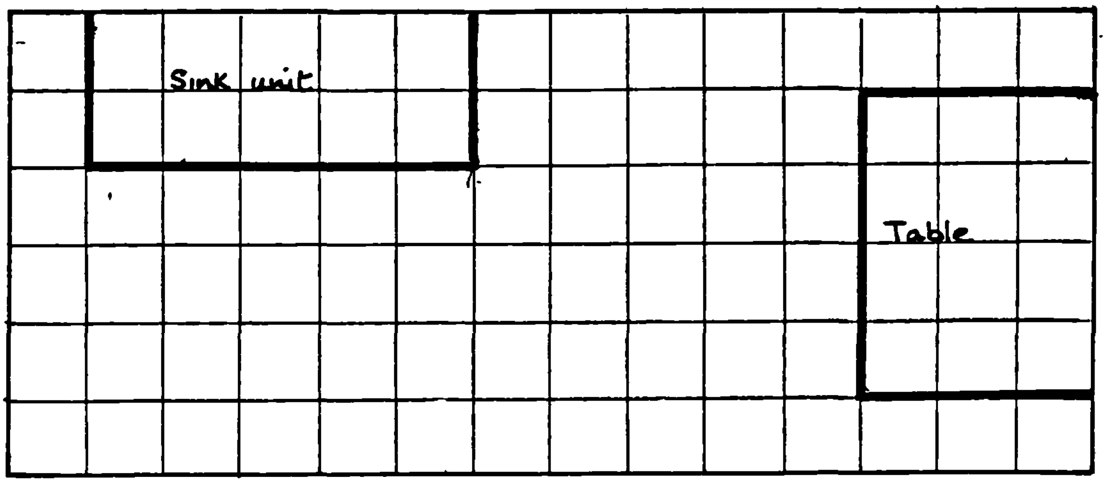
Version 1 (item used for interview)
The scale for the plan of the kitchen is 3cm to 2 metres. The refrigerator I intended putting into the kitchen is 1 metre wide. How much space should I allow for this on the plan? The window on the plan is 1.5cm above the ground, how high would a table have to be so that it exactly reached the bottom ledge of the window? How many cubic centimetres would the freezer hold if its dimensions on the plan were as follows:



The kitchen itself is 4 metres by 3 metres. How big a sheet of paper do I need for the plan?

The problem as it stands has far too many words and it is difficult to extract exactly what is being asked of the child. A second version actually gave the plan of the room and specified more exactly what was required of the child, see version 2.

Version 2



This squared paper is marked in centimetres. It represents the floor of a room. We have marked the position of the sink unit which is 2.5 metres long. What scale is being used?

What are the dimensions of the table, a) on the plan: b) actually?

a) .... b) ....

I have a gas stove which has a base measuring 1 metre by 0.5 metres. Draw this on the plan and label it. The fridge is 0.75 metres by 1 metre base. Draw this on the plan and label it.

Version 2 was also eventually rejected, the question appeared to be tapping no areas of knowledge that could not be tapped by another question. The ability to solve the various parts of the problem rested entirely on the finding of the correct scale at the start. Those children who opted for one metre for four squares were obviously going to give incorrect answers throughout, those who found the correct ratio appeared to be able to deal with the rest of the question. The form of the ratio was found in different ways. The following are excerpts from interviews with children:

Child A "2cm of paper for 1 metre. The table side is 4cm and 3cm. So the longest side is 2 metres and the shorter side is 1.5 metres."

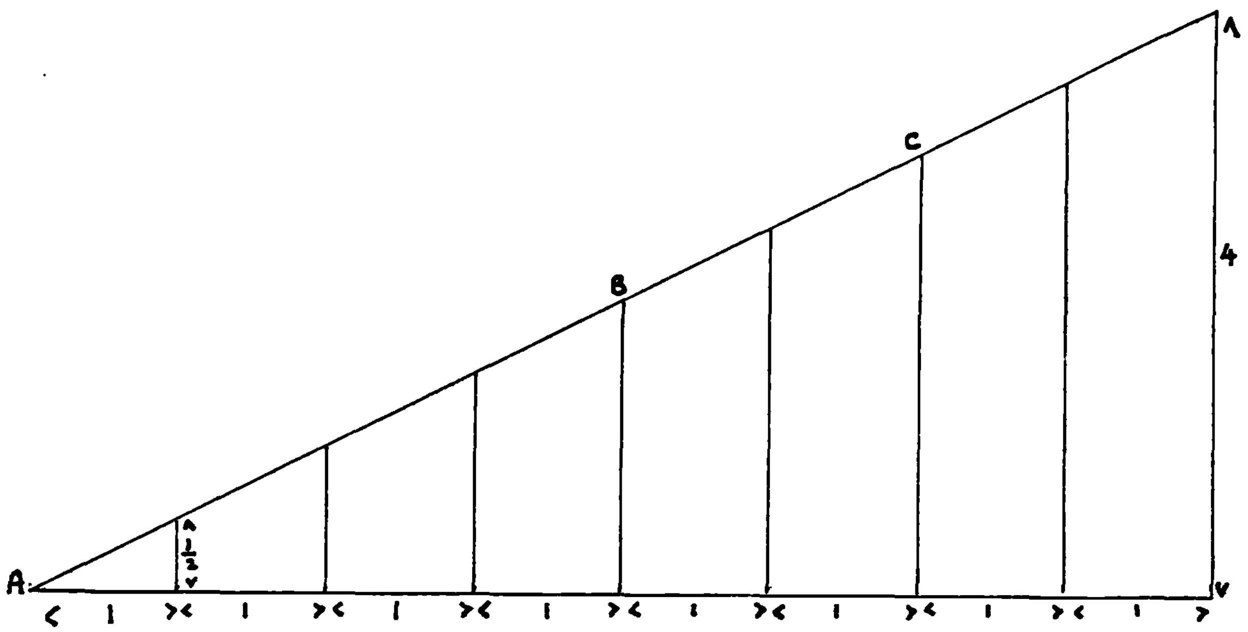
Child B "Divide 5 into 2.5 metres which is 0.5 metre into every centimetre."

Child C "Scale is 1cm to $\frac{1}{2}$ metre so $\frac{1}{2}$ metre is 1cm, another $\frac{1}{4}$ is $\frac{1}{2}$cm."

Similarity

In order to test the child's understanding of similarity of triangles, two examples were used, one consisted of a pair of triangles the other of a number of them. It was discovered early in the interviews that to most children the word "similar" meant "vaguely the same", it was only if the child had been taught the term in its technical sense very recently that he too used it in the technical mathematics sense. The word 'similar' was particularly difficult when applied to triangles and rectangles since the child's use of the word rendered all triangles similar since they were all triangles. Eventually the idea of similarity and to some extent enlargement was tested using non rectilinear figures, this will be shown in the discussion of the development of example two below.

Example 1 Version 1

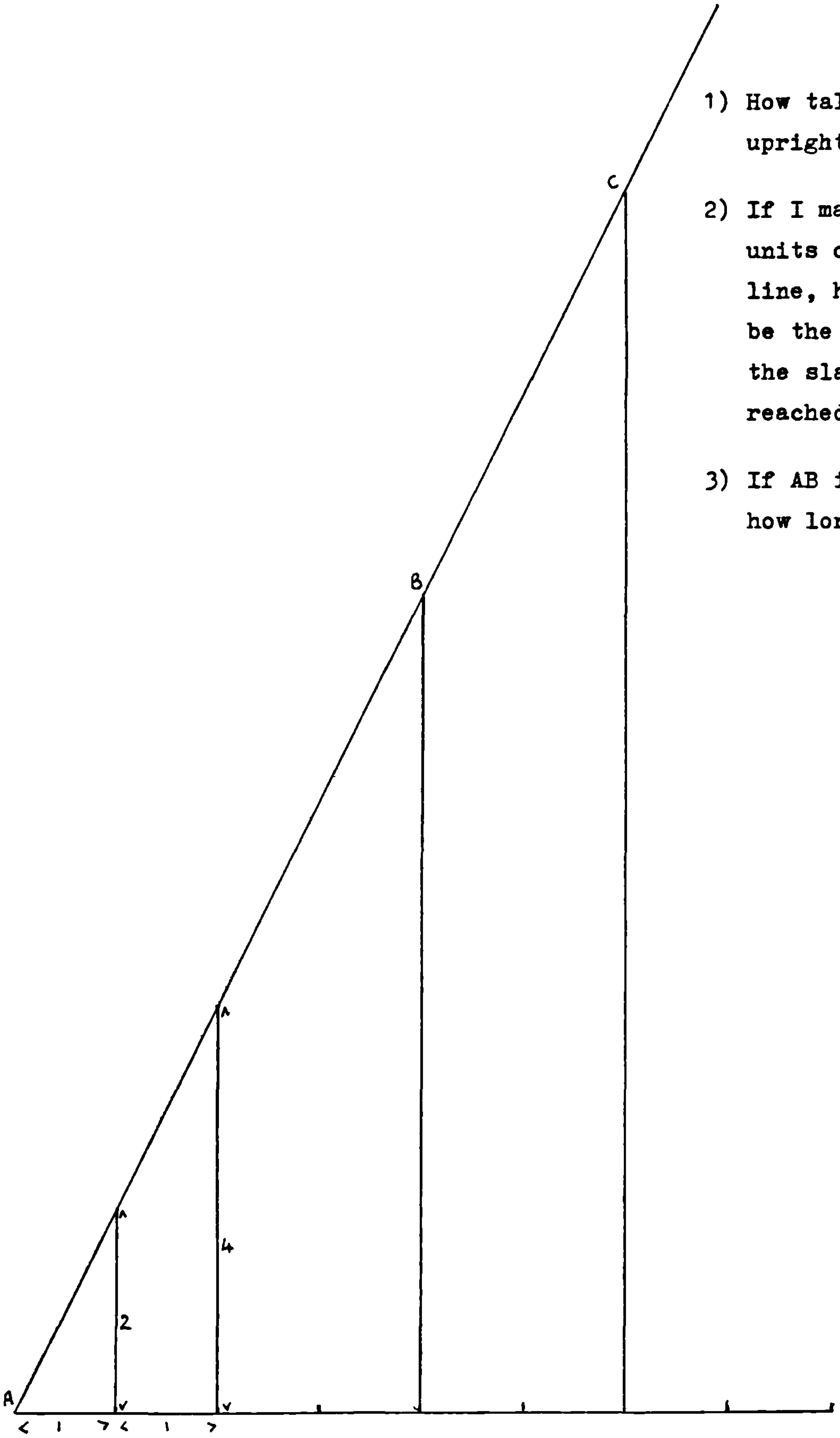


How high are the other uprights?
If I marked off 20 on the bottom line, how tall would be the upright before I hit the slant line?
If I marked off $4\frac{1}{2}$ on the bottom line, how tall would be the upright before I hit the slant line?
If AB is 4.5 which is AC likely to be?

a) $\frac{3}{2} \times 4.5$ b) 3×4.5 c) 6.5 d) none of these.

The introduction of non integral lengths added a computational factor which made the problem more difficult, the version that was used for interview therefore was simplified both in the numbers used and in the form of the diagram, see Version 2.

Version 2.



1) How tall are the two uprights shown.

2) If I marked off 20 units on the bottom line, how high would be the upright before the slant line was reached?

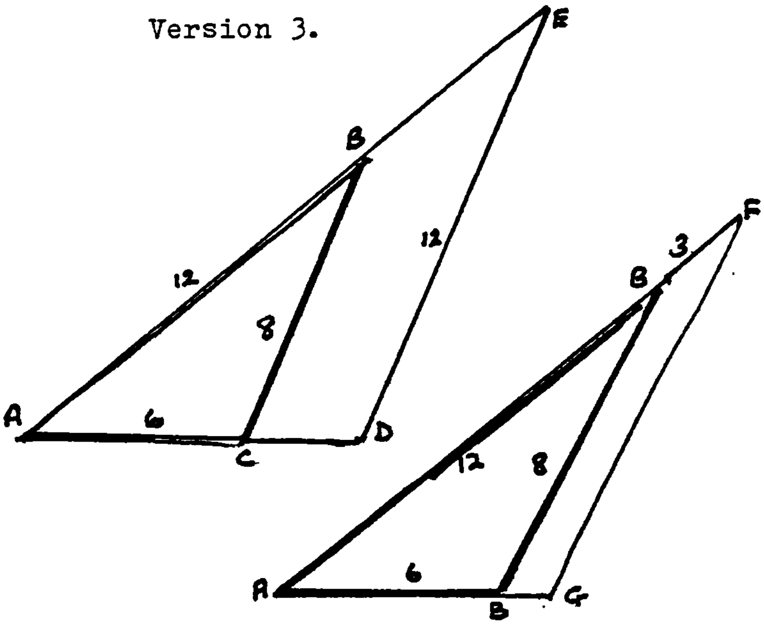
3) If AB is 6 units long, how long is AC?

In Version 2 children simply sought for the continuation of a number pattern, this was particularly easy in this case since the numbers formed the two "table", multiplication by two was seen as a short cut:

> "It goes 2, 4, 6, 8" To find the last upright the child counts on in twos and then says "Its forty, that's double."

A third version of this problem was tried but with little success.

Version 3.



AB is 12 units

AC is 6 units

CB is 8 units

DE is 12 units

How long is AE? How long is AD?

The triangle ABC is the same as above.

BF is 3 units.

How long is FG? How long is AG?

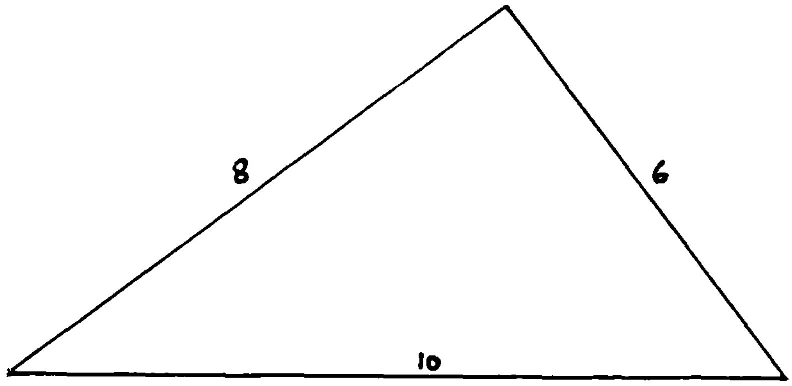
The children interviewed using this version tended to estimate the lengths by looking at the diagrams:

> David "16 for AE. AB is 12 so we need BE. The only one going up to E is DE which is 12, 4 bigger than 8. But it doesn't look right scale, looks like half, six to me." "I'd normally average and AD would be nine. CD looks like half."

Another child joined B to the figures 12 on DE and said "The 12 comes halfway down DE, its nearly the same as BE so BE must be 6."

Example two involved the comparison of two triangles, the child being asked to select the lengths of side for the second in order that it be a larger version of the first. The child was not asked to draw the second triangle having decided on the lengths of the sides, this meant that he had no way to check whether the sides he suggested did in fact produce a triangle of the same shape. To ask him to draw the triangle introduced further problems dependent on his skill in constructing triangles.

Example two on similarity - two triangles.



You are asked to make a triangle which is the same shape as this one but which can be larger or smaller. Imagine you are making the new triangle with meccano strips, you must start with the one marked 12. Which other strips would you use? Draw a rough diagram. Are there any other possible triangles you could make, still starting with the one marked 12?

<u>Meccano strips (lengths)</u>

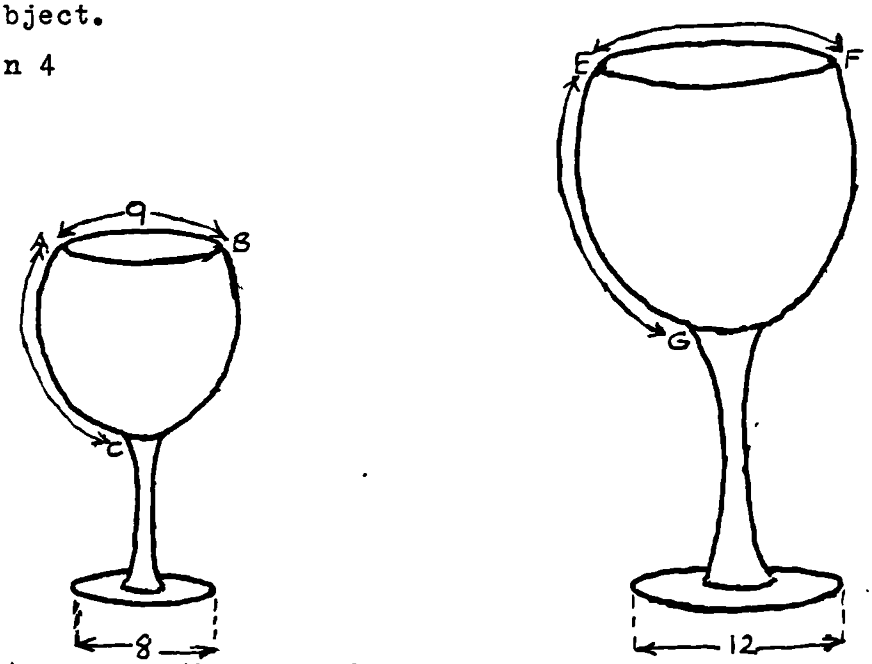
12 units	14 units
15 units	
8 units	
9 units	
20 units	
10 units	
16 units	

When the children enlarged the triangle, the realisation that to simply enlarge the sides did not necessarily preserve the angles did not occur to many, for example James aged fourteen:

> Interviewer "Which would you start with?"
> James "The largest one. Ten goes to twelve, eight to ten and six to eight. If you want to make it to the scale of that but larger, if you add 2 units onto the ten you ought to add two onto the eight and two onto the six."
>
> Interviewer "Say if you started with another side?"
> James "6 to 12, 10 to 16, 8 to 14."

The results from the triangle questions and the difficulties the children had with the questions pointed to the need for an example on similarity where the original and the enlarged figure were both given. There also seemed to be the need to provide an example where visual estimates and the finding of a number pattern were not immediately obvious. The "addition strategy" see p.31 had occurred on the triangle questions and seemed likely to continue and so would be regarded as an error worth noting in the final large scale test. The next version of the similarity question was in two forms, version four and five below. Version four was abandoned in favour of five because version four introduced the added complication of a two dimensional drawing for a three dimensional object.

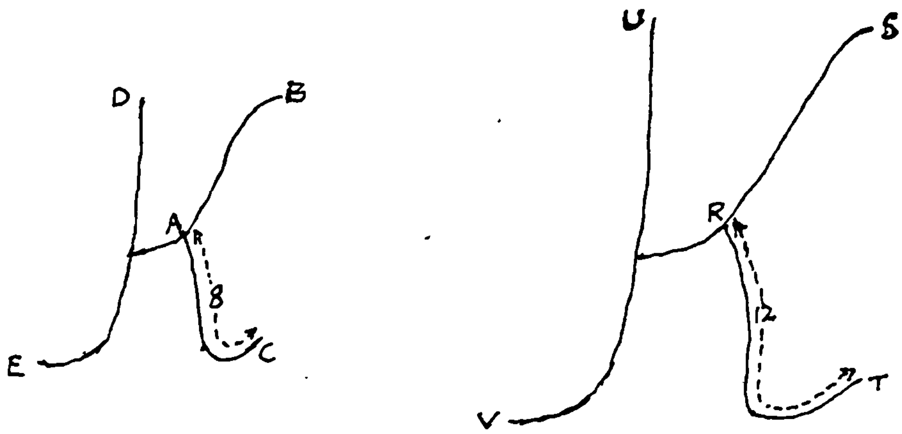
Version 4



These two pictures are the same shape, one is bigger than the other
The curve AB is 9 units. How long is the curve EF?
The Curve EG is 18 units. How long is the curve AC?

Version 5

These 2 letters are the same shape, one is larger than the other.
AC is 8 units. RT is 12 units



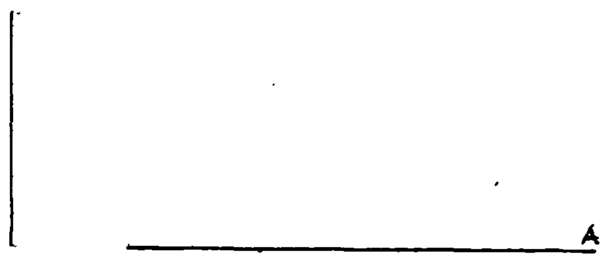
The curve AB is 9 units. How long is the curve RS?
The curve UV is 18 units. How long is the curve DE?

Enlargement

Enlargement by drawing appears in most mathematics books, the question that was used on the paper was an adaptation of one mentioned in "The Child's Conception of Space" (Piaget 1967). Since the object of the test was to test understanding it was thought that an example which did not require a pure application of skill was needed. Although in the following question it is possible to carry out the enlargement by using a centre of enlargement it is by no means obvious. The enlargement of the gap in the diagram added an extra dimension of difficulty to the question but the recognition that the gap was part of the diagram was not of course concerned with ratio. The first version of the item stated the degree of enlargement required in two cases and asked the child to find the enlargement factor in the other case.

Version 1

a) Enlarge this diagram, including the gap so that everything is double in size. A becomes A',



b) We have enlarged one line of the original diagram. Complete the picture so that you have an enlargement of all the original diagram.

c) Enlarge the following diagram so that it is triple
its size.

The second version was shorter, the word "enlargement" was omitted as too 'technical' and replaced by a more general description of what was required, the diagram with the gap still involved doubling but the child was required to find out that a ratio of 2 : 1 was required. The harder enlargement, in the ratio 5 : 3 was given in relation to a simpler diagram. The child was no longer reminded to include the gap in the enlargement of the first diagram and of course many forgot to do so, this was regarded as one of the specific errors to be noted on the large sample testing. The addition method occurred many more times on the diagram requiring 3 : 5 than on the doubling item, in fact on interview it did not occur at all on the doubling item.

Version 2

a)

Finish drawing the diagram below so that it is the same shape but bigger than this diagram.

b)

Work out how long the missing line should be if this diagram ⟶ is to be the same shape but bigger than the one above. ....cm.

The Use of a Rate.

The questions taken from Piaget and Karplus which appear below under the heading of "the work of other researchers" could be described under the present heading in that the items could be solved by finding a rate and then multiplying, the children did not necessarily do this however. Two other items on the category of "using a rate" are as follows:

> Version 1
> Three workmen send to the cafe for ham rolls, Peter ate 2, John ate 4, Brian ate 6. The bill came to 36p, how much should each pay?
>
> Peter.......... John.......... Brian..........

Version 2
In an office Mr. Adams comes in to work 2 days a week.
Mr. Brown comes in to work 4 days a week.
Mr. Carter comes in 6 days a week.
The bill for making coffee in the office for these three men is 240p.
How much should each pay for it to be fair?

Mr. Adams........ Mr. Brown........ Mr. Carter...........

The two versions of what was essentially the same question appeared on the paper used for the first interview. The ham roll problem presented little difficulty, the child usually found the cost of one roll and then multiplied.

Kim aged 13 "The number of rolls is 12. Each roll costs 3p. Two times three, four times three, six times three."

The coffee question was much more difficult, much of the difficulty occurred because the children were unable to accept that one might pay for a day's coffee without specifying how much had been drunk. This extensive interview with Janet aged twelve brings out this point.

(I is the interviewer, J the child)
J "10p, 8p, 6p
I questions - Does that come to 240p?
J "That comes to 12 and I divided 12 into that which is 20, no that's wrong". "12 is the number of days they work a week".
I Why did you bother to do that?
J "To tell you the honest truth I don't really know". "Each one works 2 more days than the next one".
I So who should pay the most?
J "Mr. Carter".
I Any relationship between Adams and Brown?
J "Mr. Brown works 2 days more than Mr. Adams". "It depends on how many times they have coffee a day".
I What can we assume?
J "They have one cup a day". "That has 2 cups of coffee a day, 4, 6, that makes 12 cups of coffee altogether in a week, 12 cups of coffee a day in a week".
I Can you have that?
J "12 cups a week then and one each day". "It must be 20p a cup of coffee".
I You make the assumption they had 1 cup a day?
J "Oh yes, they might have 2 cups a day".

I Would that make a difference?
J "Yes".
I Do you think it would help if we knew how much a cup of coffee cost?
J "It wouldn't really help".
"He might have 3 and he might have 1".
I I think we must assume that when they are in, they drink the same amount.
J "If each man had 2 cups of coffee each day, that's 24 cups of coffee, that's 10p".
I How much would Mr. A pay?
J "20p".
I But he's in for 2 days.
J "40p; 80p; 120p"
Checks that sum is 240p.
I Say if I told you they had 4 cups each a day. Would that make a difference?
J "Yes. No, 4 cups a day - 80p".
I How much would they cost?
J "2 cups cost 20, 4 cups would cost"
"No you wouldn't have different answers. Cups of coffee wouldn't be so expensive".
"It depends on total amount they have to pay".

It was thought that the question should involve the finding of a rate not as obvious as the cost of one object but that it should not have embedded in it the suspicion that yet another rate was involved (as occurred in the coffee question). The version used on the wide scale survey was therefore the following:

> In an office Mr. Adams comes in to work 2 days a week.
> Mr. Brown comes in to work 4 days a week.
> Mr. Carter comes in 6 days a week.
> The bill for lighting the office for these three men is 240p.
> How much should each pay for it to be fair?
>
> Mr. Adams........ Mr. Brown........ Mr. Carter........

A very much harder question involving currency rate of exchange was abandoned after a few interviews. The following not only involved multiplication but also percentages and fractions. The need to keep the context of the question as simple as possible was very important and so any extra computation or complication of language was avoided. Technical terms were also avoided, the problem being finally posed in terms of what was fair rather than in terms of proportion.

Currency question

a) I am going to Switzerland on holiday. I change £15 into Swiss francs. The rate of exchange is 8 Swiss francs to £1. How many Swiss francs do I get for my £15?

£15 = ...Swiss francs

b) On my return I have 34 Swiss francs left. I change these back into pounds. What do I obtain for them?

34 Swiss francs = ...£

Doubling and halving

The enlargement by drawing involved the doubling of the length of a line, a question was needed whereby the child had to double or halve without necessarily finding a rate as in the electricity question. The following item involved a recipe.

After interview it was made shorter and the move from eight to six in the first version was made more gradual in the last version by asking for the amount for four people and then six people's requirements. This intermediate step occurred in the reasoning of many of the children as will be described later.

Version 1 Onion Soup Recipe for 8 persons
8 onions
1 pint water
4 chicken soup cubes
2 dessert spoons butter
½ pint cream
I have only six people to feed and I do not wish to waste any soup. How would I alter the recipe to give the same type of soup?
Onion Soup recipe for 6 persons.

.... Onions
.... Water
.... Chicken cubes
.... Butter
.... Cream

Version 2
Onion Soup Recipe for 8 persons
8 onions
2 pints water
4 chicken soup cubes
2 dessertspoons butter
½ pint cream

a) I am cooking onion soup for 16 people.
How many onions do I need? ....
How much cream do I need? ....

b) I am cooking onion soup for 4 people.
How much water do I need? ....
How many chicken soup cubes do I need? ....

c) I am cooking onion soup for 6 people.
How much water do I need? ....
How many chicken soup cubes do I need? ....
How much cream do I need? ....

Version 3.
Onion Soup Recipe for 8 persons
8 onions
2 pints water
4 chicken soup cubes
2 dessertspoons butter
$\frac{1}{2}$ pint cream

a) I am cooking onion soup for 4 people
How much water do I need? ....
How many chicken soup cubes do I need? ....

b) I am cooking onion soup for 6 people.
How much water do I need? ....
How many chicken soup cubes do I need? ....
How much cream do I need? ....

The final version involved one ingredient quoted in terms of a number, one which was quoted as a unit of measurement and the third which was also in units of measurement but was given initially as a fraction.

Items not requiring ratio

A series of items which common sense methods would show did not need ratio were used on the first interviews with Grammar school children. The trap was to assume a constant rate and in fact many of these able children did make such an assumption, it was only after the interviewer had probed further that they saw the fallacy in their argument. The assumption appeared to be part of what was expected in a mathematics question and the children felt that they were being tricked. The final paper had to be in paper and pencil format so there would be no opportunity to urge the children to think again, the following items were therefore omitted. The attitude of the children is illustrated by the two interviews which follow the questions:

Questions
Tick which of the following are obviously true.
If not true state why.

a) Roger Bannister was the first man to run a mile in 4 minutes. He ran 5 miles in 20 minutes.
....

b) A Vienna loaf costs 12p, the shopper buys three, she pays 36p.
....

c) For one cake I use 10 ounces of flour, for two of these cakes I use 20 ounces.

....

d) A duster takes 40 minutes to dry, the three dusters I have hung on the line should take 2 hours to dry.

....

Interview 1. part a) in above set of questions
(Interviewer K; Child A)

A "Yes. If he ran 1 mile in 4 min. 20 min. is 5 times 4 min".
K Say if it was you.
A "I'd get tired".
A "Mathematically its true, physically its not".
K What do you mean by mathematically its true?
A "If it was a vehicle it would be true. People get tired".

Interview 2 (Interviewer K, Child B) parts a), b), c), d) as above.

a) B True
K Question - say if you were doing that?
B "I was doing it mathematically not logically". "If you were doing it mathematically you don't take notice of how much energy you need".
b) B "If Vienna loaves were different weights and different prices, it would be different". Correct.
c) B "Because they're both the same cakes". Correct.
d) B "Because if you hang them all on the line at the same time, they'd all dry at the same time. One takes 40 mins so three must take 40 mins". Correct

Ratio needing an intermediate step

The intermediate step requirement was an attempt to provide a ratio question which was more complex than those already mentioned. The actual ratio required was fairly simple 2:1 or 3:2 but the child had to recognise that a third dimension was required for each. In the following question on chemical compounds the connection between the two amounts for which the ratio was required was given in terms of a third substance which did not appear in the answer. Only two versions were tried, the final version was worded differently to the first version. The word "parts" seemed to cause no difficulty on interview so was retained.

Version 1
In a particular chemical compound there are
1 part mercury to 5 parts copper
3 parts tin to 10 parts copper
8 parts zinc to 15 parts copper

State a relationship between the mercury and tin contents and between the zinc and tin.
.... parts mercury to ..... parts tin
.... parts zinc to ..... parts tin

Version 2
In a particular metal alloy there are
1 part mercury to 5 parts copper
3 parts tin to 10 parts copper
8 parts zinc to 15 parts copper
You would need how many parts mercury to how many parts tin?
..... parts mercury to ..... parts tin
You would need how many parts zinc to how many parts tin?
..... parts zinc to ..... parts tin

Percentages

The topic of percentages appears in the secondary school syllabus but is often regarded by the child as an exercise in multiplying fractions. In order to retain the basic idea of a ratio a:100, the final version of the test gave an explanation of the percentage symbol in terms of "per hundred". Each percentage question leant itself to this interpretation in that the numbers given could be easily thought of in terms of a certain number of hundreds and halves of hundreds. The first version assumed that the child knew the meaning of percentage and further provided a multiple choice question involving large numbers and a method of working out the answer. Version One was as follows:

a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ....
b) 6% of children in a school have free dinners. There are 250 children in the school.
How many have free dinner? ....
c) The newspaper says that 24 out of 800 'Avenger cars have a faulty engine.
What percentage is this? ....
d) 35 per cent of all adults read a newspaper. If a town has 25,300 adults, how many newspapers would one expect to be sold there?

(i) $\frac{35 \times 100}{25,300}$ (ii) $\frac{25,300 \times 35}{100}$ (iii) $\frac{25,300 \times 100}{35}$

(iv) $\frac{35 + 25,300}{100}$ (v) $\frac{25,300}{100 + 35}$ (vi) 35×253

The multiple choice format for question (d) was not very successful, on interview it soon became apparent that the methods given did not correspond to those normally used by the children and in order to choose one of them the child was reduced to working out the answer and then working out each of the options. Alternatively the child opted for one of the methods but was either unable to explain why or gave reasons not connected with the computation. James aged fourteen knew why he opted for method (vi) "Its the simplest way. It has smaller numbers". Examples of children who worked out the question themselves or provided alternatives to the six methods provided were:

Mark "You want really $\frac{35}{25,300} \times 100$

Tim " $253\overline{)25300}$ with quotient 100 I've got to get my 100 then".

Patrick "3500 per 10,000
3500 " "
1750 per $\frac{1}{2}$(10,000)
105 3 times 35".

Version Two was constructed so that a minimum of computation was required but flexibility in the understanding of percentage was needed.

Version 2

% means <u>per cent</u> or <u>per 100</u>, so 3% is 3 out of every 100.

a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ....

b) 6% of children in a school have free dinners. There are 250 children in the school.
How many children have free dinner? ....

c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.
What percentage is this? ....

d) The price of a coat is £20, in the sale it is reduced by 5%; how much does it now cost? ....

Items Taken from Other Researchers:

Piaget

In "Epistemologie et Psychologie de la Fonction (1968)" Piaget presented a problem concerning three fish which were, five, ten and fifteen centimetres in length, they were fed according to their length so that C(15) was fed three times as much as A(5) and B(10) was fed twice as much as A. The problem was given in two forms, one where the fish were fed a number of discrete objects (meatballs) and the other where the food was still proportional to the length of the fish but was in fact biscuits of different length. The method of presentation was by interview (the children were aged 5-9 years). Fish A(5), Fish B(10) and Fish C(15) formed the basis for discussions; one meatball was placed by A and the child was asked how much one should give to B and C; then B received four meatballs and the child was asked how much should A and C receive and finally C received nine meatballs and the child was required to find how many for A and B. The same procedure was followed using lengths of biscuit, the numbers being as before. The version on the Ratio paper was in two parts, the first very close to the meatball problem above, the second required lengths rather than discrete objects as above but the lengths of the fish were changed to ten, fifteen and twenty-five centimetres thus providing an example of non-integer multipliers. Biscuits and meatballs were replaced by fishfingers and sprats respectively and the fish were designated "eels". The first version using food which was composed of discrete objects retained the numbers 1, 4 and 9 sprats but these were changed on the subsequent version. Lines 5, 10 and 15 centimetres long were drawn to represent the eels:

First Version

There are 3 eels A, B and C in the tank at the Zoo.

15 cm long A

10 cm long B

5 cm long C

The eels are fed sprats, the number depending on their length.

a) If C is fed one sprat, how many sprats should B and A be fed to match?

.........B ....A

b) If B eats 4 sprats, how many sprats should A and C be fed to match?

.........A C

c) If A gets 9 sprats, how many sprats should B and C get to match?

.........B C

Piaget identified the strategy of adding on just one more meatball for a larger eel, as stage II (age 6-8) "a A; elle donne 2 a B et 3 a C parce que (B) est le poisson moyen et (C) est le plus grand". In the first version of the item (where C is fed one sprat) the correct answers 2, 3 may be arrived at by doubling and trebling or by this stage II addition method. Fiona for example gave 1, 2, 3 for part a) then 3, 4, 5 for part b) and 9, 10, 11 for part c). It was not until she attempted part b) that one could decide that she was adding on one.

In order to make the first part the easiest, it was decided to change the amount given to the smallest eel to two sprats, this now made the second part of the question identical to the first with answers 2, 4, 6. The second part was therefore changed so that B received twelve sprats (six sprats would have made the question identical to part c). The question was also shortened so that the child no longer needed to find the amount for each eel on every part of the question. The version that appeared on the final paper was as follows:-

> There are 3 eels A, B and C in the tank at the Zoo.
>
> 15 cm long A
>
> 10 cm long B
>
> 5 cm long C
>
> The eels are fed sprats, the number depending on their length.
> If C is fed two sprats, how many sprats should B and A be fed to match?
> B.........   A..........
> If B eats 12 sprats, how many sprats should A be fed to match?
> A..........
> If A gets 9 sprats, how many sprats should B get to match?
> B.........

Piaget stated that the eel question using lengths of biscuit was solved rather later than that involving discrete objects, stage II appearing at age nine. It was decided to make the comparable question on the Ratio paper more difficult by giving eel lengths which differed by factors other than two and three. The amounts of food given to the eels therefore also had to be changed so that the finding of a rate (so much for five centimetres of eel) would not involve a fraction. Version one required the amounts for two eels in each of three situations, the third part however gave identical answers to the first and was therefore changed on subsequent versions. In the final version the eels were drawn as straight lines (to scale) since children on interview "stepped off" along the line segments to enable them to see the problem more clearly. Again the final version did not require the answers for six eels but only four.

Version 1
As an experiment 3 other eels, X, Y and Z are fed with fish fingers. The length of the fishfinger depending on the length of the eel.
X is 10cm long.
Y is 15cm long.
Z is 25cm long.
If X has a fishfinger 2cm long, how long should the fish-fingers given to Y and Z be? Y........ Z........
If Y has a fishfinger 9cm long, how long should the fish-fingers given to X and Z be? X........ Z........
If Z has a fishfinger 5cm long, how long should the fish-fingers given to X and Y be? X........ Y........

Final Version
Three other eels, X, Y and Z are fed with fishfingers, the length of the fishfinger depending on the length of the eel.

25 cm long Z

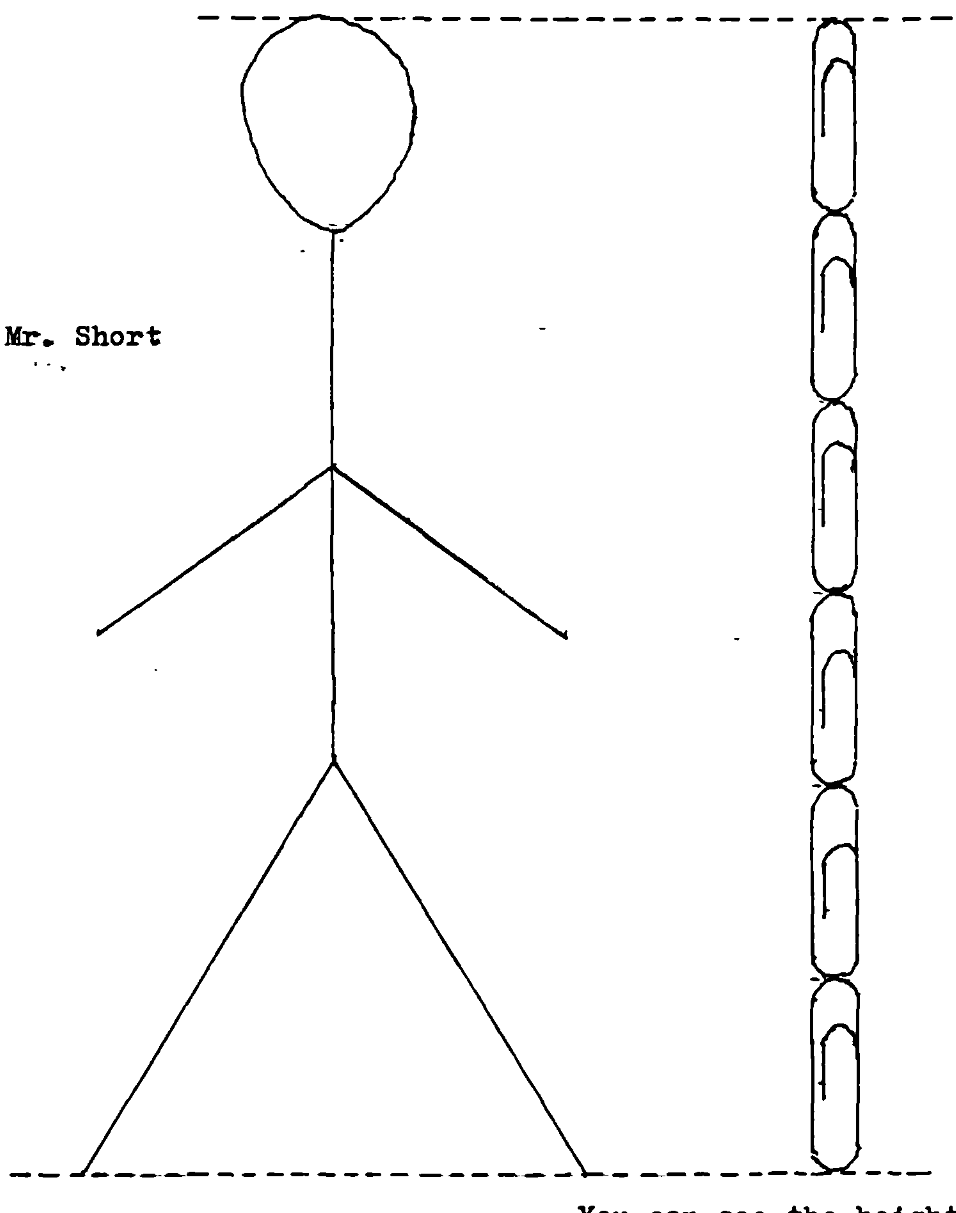
15 cm long Y

10 cm long X

If X has a fishfinger 2cm long, how long should the fish-finger given to Z be? Z........
If Y has a fishfinger 9cm long, how long should the fish-finger given to Z be? Z........
If Z has a fishfinger 10cm long, how long should the fish-fingers given to X and Y be? X........ Y........

Karplus

The research of Elizabeth and Robert Karplus has been discussed in Chapter Two. One of the items they used in their seven nation study was that of Mr. Short and Mr. Tall. Their research was by paper and pencil test and involved the measuring of the pin man by paper clips, the child being provided with an adequate amount of these. Since the version to be used in this study was not in the researcher's presence and since the size of paper clips varies so much, it was thought less cumbersome to draw the paper clips, thus avoiding the possibility of children estimating the length of the pinman as a fraction of a paper clip. The second unit of measure in the Karplus study was a button, this was replaced by a matchstick in the present study. The paperclips which appeared beside the pinman were $2\frac{1}{2}$cm in length and therefore not a convenient number of centimetres for calculation using the actual height of the pinman. The version in the final survey did not differ appreciably from earlier ones and appears **at the end of this chapter**



You can see the height of Mr. Short measured with paper clips.

Mr. Short has a friend Mr. Tall. When we measure their heights with matchsticks:
Mr. Short's height is four matchsticks
Mr. Tall's height is six matchsticks
How many paper clips are needed for Mr. Tall's height? ....

Pilot Study - Class Tests and Discrimination Analysis

After a number of children had been interviewed using the Ratio test as the interview instrument, items were deleted and others introduced. Finally a version was arrived at which seemed to be understood by the children, to have a range of examples spanning different difficulty levels and which would fit into a mathematics lesson of forty-five minutes. The printed papers were then tried on two hundred children in London secondary schools. The children were given the paper by the researcher and told that they were taking part in educational research, the researcher encouraged them to raise their hands if there was any word or expression they did not understand. The papers were then marked according to the marking scheme described later. It was hypothesised that difficult items i.e. those with low facility, would be successfully completed by those children with high total scores, those with low total scores should have acquired those scores on the easy items. Any items which appeared to be successfully completed by children with either high or low total scores deserved to be looked at again since they were not discriminating. Each child within the sample of two hundred was ranked according to total score and his performance on each item recorded, see Appendix 2 .
The lighting question appears with three separate answers but was later marked as one question, 4b (see final paper in this chapter) does not appear at all since two versions of this were given in the class tests. As can be seen, some items were solved randomly by a few children but on the whole the most difficult items were solved only by the children with the highest total scores. An attempt was made at this stage to group items simply on the basis of "plateaus" occurring on the diagram.

Facility levels for the items were found firstly for the total number of children from four different schools and then for three second year classes within one school (Appendix 3). The three classes although the same age, were very different in their ability to solve the items on the ratio paper. Next, three different age groups were compared, twenty-nine fourth formers, seventy-five third formers and seventy-five second formers, the year totals being made up from different schools, the results appear in Appendix 3. There was no obvious distinction between year group performance although the fourth year tended to do rather better. It was apparent that on some questions the year groups were very close in performance and on others there was considerable disparity. On the second eel question the third year performed rather worse than the second year, the sample for each year however was not representative of the entire I.Q. range. An attempt was made even at this early stage to gain some insight into the child's performance on the ratio questions and his cognitive level as defined by Piaget; with this in mind answers on the eel question were labelled with a cognitive level and in addition another Piagetian item from "Epistemologie de la Fonction" (1968) was given in class test format to two of the classes see **Appendix 4**. This second question involved the sharing of numbers e.g. share thirty sweets between two boys so that one has six more then the other. It was not very successful in class test format as children who committed the error of halving and adding (30 ÷ 2 = 15, add 15 to 6 and subtract 6 from 15; answer 9, 21) on the first part tended to do all the other parts of the question in similar ways. On the final form of the Piaget Class Task (see later) a check was built into the question in order to force the children to think again. It was apparent that children who were performing at level IVA on the sharing task were using the addition strategy on the Ratio questions, this was therefore not a strategy adopted only by the children with a low cognitive level. The few children at level IVB (early formal) on the sharing question did not use the addition strategy.

The class tests therefore provided evidence of words that needed to be changed (the enlargement of the open figure question was changed), facility levels, ideas on the discrimination of items, common incorrect answers and the fact that performance did not appear to be completely tied to age level. The Piagetian items were insufficient in number to ascertain cognitive levels in order to match them against performance on the Ratio items. The facilities of the items and the available strategies for solution of each problem appear in Table 5. The strategies are further discussed in the next chapter.

Table 5:

Items in Order of Difficulty from the Class Tests
(Pilot Study) n = 200

Strategies available and difficulties in the items are described, the item numbers refer to the written test which follows.

Item	Part	Percent	Strategies Available
1a	1	(94)	Halving, rate per person
1a	2	(92)	Halving, rate per person
3a	1	(87)	Rate per 5cm then doubling
8	i	(84)	Substitution in the example given
1b	1	(77)	Half, half again and add; rate per person and multiply by integer; multiply by fraction
3a	2	(75)	Adding two amounts already known; rate per 5cm multiply by 3
1b	2	(70)	Same as 1b. 1
4a	1	(62)	Find that doubling is needed and then multiply by two
2	B	(51)	Recognise rate is needed, find rate per day,
	A	(48)	multiply by 2, 4, 6 or find half for C and
	C	(47)	divide remainder between A and B (half and whole)
3a	3	(44)	Find rate for 5cm and then add or multiply by 3
3a	4	(44)	Same as 3a. 3
3b	1	(43)	Y is half again then add. Take the amount once again and half again. Find rate for five, add and then take half. Rate and multiply by fraction
8	ii	(42)	Technique taught or 6% taken twice and a half
4a	Vert gap	(31)	Notice double and double length. Technique using centre of enlargement. Distractor in that gap is not noticed as part of the diagram

Item	Part	Percent	Strategies Available
4a	Horiz gap	(30)	As above.
6a		(29)	Recognise copper is needed, then double
8	iii	(29)	As 8 . ii
5		(25)	Recognise ratio, use match to paperclip rate. Take six and half of six. Multiply by fraction
3b	3	(19)	Rate for 10cm and then build up, Rate and multiply by fraction
3b	2	(18)	as above
3b	4	(18)	as above
8	iv	(17)	As 8 . ii Distractor - forget to subtract, mix units
1b	3	(14)	Half and half again. Fraction has to be halved
7b		(9)	Recognise ratio needed, finds rate and step down by $1\frac{1}{2}$
6b		(8)	Notice copper needed, find rate per five and multiply by 3
7a		(7)	As 7b, step up or multiply by $1\frac{1}{2}$.

Final Test

TEST R

1. Onion Soup Recipe for 8 persons

   8 onions

   2 pints water

   4 chicken soup cubes

   2 dessertspoons butter

   ½ pint cream

   a) I am cooking onion soup for 4 people

      How much water do I need? ......Variable 1

      How many chicken soup cubes do I need? ......Var. 2

   b) I am cooking onion soup for 6 people

      How much water do I need? ......Var 3

      How many chicken soup cubes do I need? ......Var 4

      How much cream do I need? ......Var 5

----

2. In an office Mr. Adams comes in to work 2 days a week.

   Mr. Brown comes in to work 4 days a week.

   Mr. Carter comes in 6 days a week.

   The bill for lighting the office for these three men is 240p.

   How much should each pay for it to be fair?

   Mr. Adams .... Mr. Brown .... Mr. Carter ....

   Var 6

3a) There are 3 eels A, B and C in the tank at the Zoo.

15 cm long A

10 cm long B

5 cm long C

The eels are fed sprats, the number depending on their length.

If C is fed two sprats, how many sprats should B and A be fed to match?

B ..Var.7.. A ...Var.8.

If B eats 12 sprats, how many sprats should A be fed to match?

A .Var.9...

If A gets 9 sprats, how many sprats should B get to match?

B .Var 10..

3b) Three other eels, X, Y and Z are fed with fishfingers, the length of the fishfinger depending on the length of the eel.

25 cm long Z

15 cm long Y

10 cm long X

If X has a fishfinger 2 cm long, how long should the fishfinger given to Z be? Z ...Var 11.

If Y has a fishfinger 9 cm long, how long should the fishfinger given to Z be? Z Var 12

If Z has a fishfinger 10 cm long, how long should the fishfingers given to X and Y be? X .... Var. 13 Y .... Var 14

4a

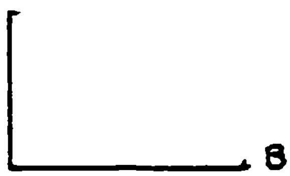
Var. 15

Finish drawing the diagram below so that it is the same shape but bigger than this diagram.

Var 16

Var 17

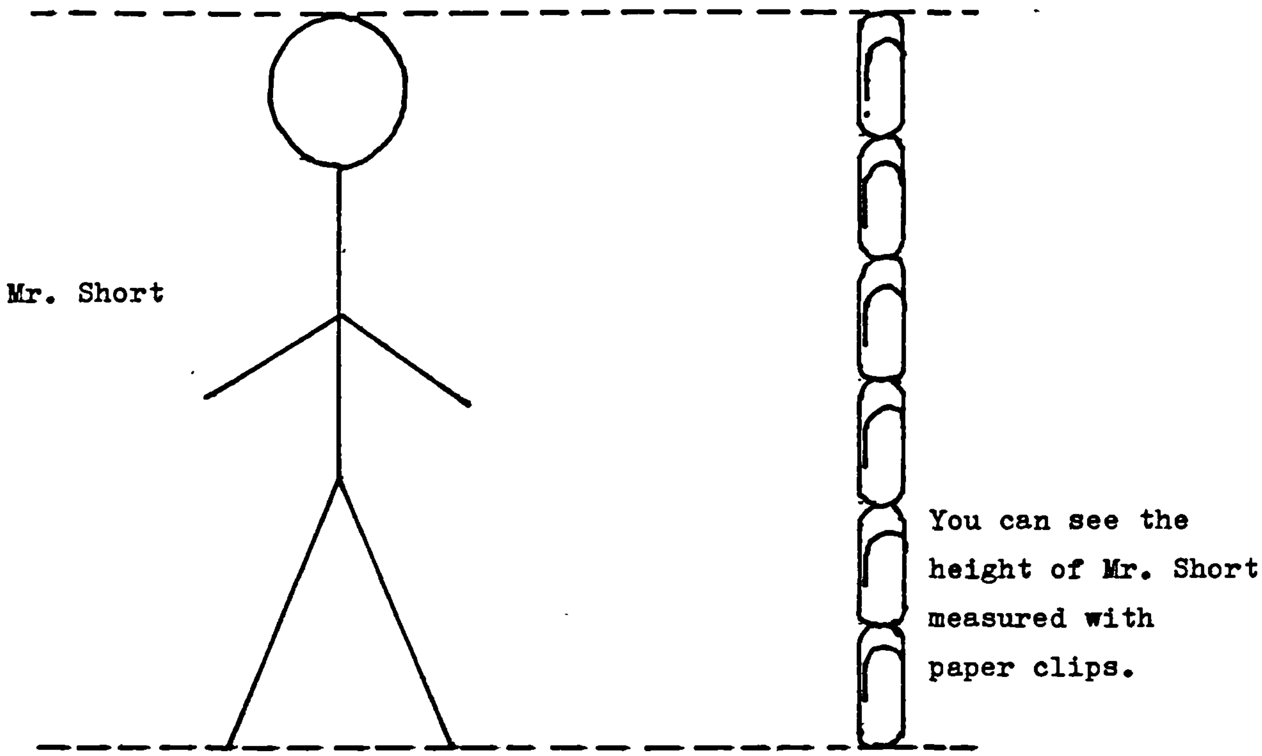
4b



Work out how long the missing line should be if this diagram is to be the same shape but bigger than the one above

Var. 18

5



Mr. Short has a friend Mr. Tall. When we measure their heights with matchsticks:

Mr. Short's height is four matchsticks

Mr. Tall's height is six matchsticks

How many paper clips are needed for Mr. Tall's height? ....

Var. 19

6. In a particular metal alloy there are

1 part mercury to 5 parts copper

3 parts tin to 10 parts copper

8 parts zinc to 15 parts copper

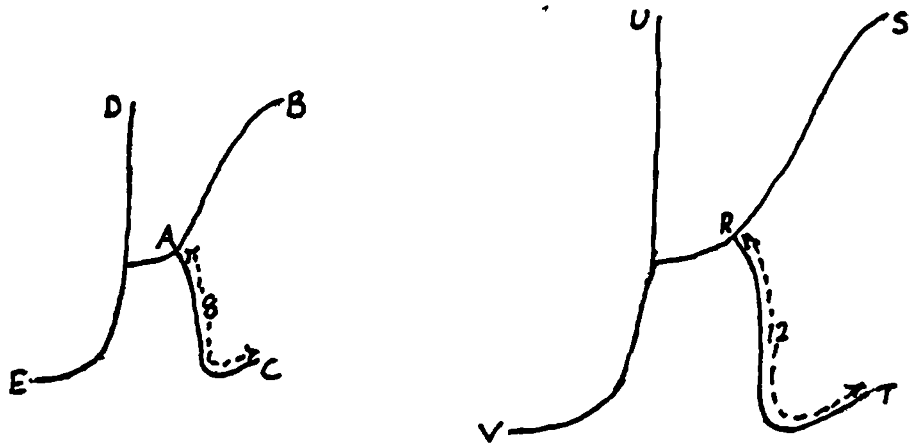
You would need how many parts mercury to how many parts tin?

...... parts mercury to ...... parts tin Var. 20

You would need how many parts zinc to how many parts tin?

...... parts zinc to ...... parts tin Var. 21

7. These 2 letters are the same shape, one is larger than the other. AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS? Var.22
The curve UV is 18 units. How long is the curve DE? Var.23

8. % means <u>per cent</u> or <u>per 100,</u> so 3% is 3 out of every 100

a) 4 children out of the hundred on the school trip forgot to bring their lunch.

What percentage is this? ....Var 24

b) 6% of children in a school have free dinners. There are 250 children in the school.

How many children have free dinner? ....Var 25

c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.

What percentage is this? ....Var 26

d) The price of a coat is £20, in the sale it is reduced by 5%; how much does it now cost? ....Var 27

CHAPTER FOUR

The Implementation of The Survey

The construction of the test instrument and the pilot testing of it was completed in the spring of 1976. The wide scale testing was to take place in the summer of 1976. In this chapter two aspects of the study which can be regarded as preliminary to the actual testing are discussed. There is a discussion of the sample chosen, both as it was intended and also when subsequent changes had to be made. The problem of analysing the data obtained from the testing is also introduced with reference to available methods, although the actual analysis based on the data obtained from the testing on the ratio paper appears in the chapter detailing results.

Sample Chosen for Testing, 1976 and 1977

In order to obtain a sample of children which could be regarded as representative of the British child population, it was decided to use schools throughout England in both urban and rural communities. The representative nature of the sample was to be based on the IQ range so that the total sample for any one year (age range) tested would mirror the IQ spread on a standardised test. The papers were to be given in schools and so teachers were to be involved in the administration of the test. Consequently each time a members of the mathematics team gave a talk to a group of teachers, she asked for schools to volunteer a) children for testing and b) the assistance of the teachers. Many teachers did volunteer their pupils for testing, so many in fact that not all were used. The sample is consequently from schools where teachers volunteered their help and where those same teachers attended in-service courses or meetings arranged by the local mathematics adviser.

The IQ test selected was a non-verbal one issued by the NFER, it had already been used by the Science team of the C.S.M.S. Project. The short form of the Calvert non-verbal test took half an hour to administer and was stated as suitable for children aged 10.08-11.07. Standardisation could be calculated for children up to the age of thirteen but it tended to militate against the older children in that it produced a cut off of the higher IQ scores at that age. No IQ test of the same type was found for children older than thirteen. This meant that the IQ scores for children other than the second year would not probably be available, although two schools did have this information on their third and fourth years. It was decided therefore that the IQ scores of the second year would be used to obtain a picture of the spread of intelligence in any one school. If the staff at that school stated that the status of the school had not changed e.g. it had not changed from secondary modern to comprehensive, the basis for selection of pupils and the catchment area had not changed, and that they could say the second year was very well matched to the third and fourth, it was deemed that the spread of IQ in the older age groups was the same as that in the second year. If the status had changed then only the second year was used for testing in that school. All second years in any school used in the sample were therefore tested with the Calvert IQ test.

In 1976 eight schools were asked to test their second years on Ratio. These were five secondary comprehensives, two grammar schools and one middle school. After the testing had taken place it was discovered that two of the schools had omitted to test a class and so these children had to be omitted from the expected sample. Using the class lists for the second year in each school the pattern of IQ in that school was found by assigning the children to thirteen IQ groups from -72 to 128+, each group having a range of five points. The totals for all the schools in each IQ group was then found and compared with the expected number in each IQ group from the standardisation of the Calvert test. The obtained totals were compared with the expected totals using the Kolmogorov Smirnov Technique (Siegal 1956, p.47), the results appear in Appendix 5. It was found that the children from the second grammar school were not needed, in that the totals were adequate without them, therefore their Ratio results were omitted from the analysis. Appendix 6 shows the second year sample.

The IQ of the third year was not known so the pattern of IQ and the number of children in each IQ group of the second year was used in order to find the closeness of the sample to the normal distribution. The third years tested came from five comprehensive schools and one grammar school. One school omitted to test its remedial group so the total for the school was adjusted accordingly. One school did not test its children at the same time as the other schools but after the long summer vacation so the results of its children were omitted from the sample analysed. The third year IQ distributions appear in Appendix 7.

The fourth year sample was matched in exactly the same way as described above. One school split each year group in half for administrative reasons so half of the fourth year was tested. It was found that more children in the high IQ range were needed to approximate to the normal IQ distribution, therefore a second grammar school was approached (which had the IQ scores of these children when they were 12 years old) and selected girls' results were used in the analysis. The fourth year sample was taken from five comprehensives and two grammar schools, the distribution of IQ appears in Appendix 8.

None of the schools were in London, seven were urban schools and the rest were in Hertfordshire and Gloucestershire. Four schools tested children in two age groups and the rest in just one. Since this study was part of the C.S.M.S. survey the children each did two test papers, see below in table 6.

Some details of school organisation and the organisation of the mathematics classes appear in Appendix 11.

Table 6. The Approximate Number of Children who Took Two or More Tests, 1976.

	2 tests, or more						3 tests, or more				4 tests
	Algebra Ratio	Algebra Vectors	Algebra Graphs	Ratio Vectors	Ratio Graphs	Vectors Graphs	Algebra Ratio Vectors	Algebra Ratio Graphs	Algebra Vectors Graphs	Ratio Vectors Graphs	Algebra Ratio Vectors Graphs
2nd year	670	50	250	50	160	50	50	50	50	50	50
3rd year	680	730	250	600	320	360	530	250	190	260	180
4th year	500	360	90	310	20	380	250	20	90	20	20
All years	1850	1140	590	960	490	790	820	310	320	320	250

Note: The figures have been rounded up to the nearest 10.

Each school was visited and the teachers were asked to give the test in their mathematics lessons. The time limit for the test was approximately forty minutes but teachers were told that the test could be given in a shorter time, the intention was that the teachers should not be too harsh either with time allowance or with the reading ability of the children, although few would indeed read the questions to the children. When the test was finally given, the teachers whose classes were being tested were asked to answer certain questions on the paper, the question sheet appears in Appendix 9. All the testing of the 1976 Ratio sample took place in June and July of 1976 when the children were very nearly at the end of the school year. The number of children who actually did the test from each school appears as Appendix 10., the total number of children tested in 1976 was 2257.

In 1977 the sampling procedure was rather different in that C.S.M.S. had some thirteen papers to test, it was therefore impossible to test entire school year groups on any one topic paper. Consequently the decision was taken that each age group tested should be selected from four to six schools. The six schools were matched in pairs so that each pair together provided the normal spread of IQ. Then each school was asked to do four test papers with each class, the selection for each paper was done by taking every fourth child on the class list. Each test paper was sent to the school, already labelled with the name of the child who was to attempt it. It was hoped that each child would complete two test papers so the four papers done by any school had to comply with the overall matching pairwise.

The letters sent to the teachers explaining this arrangement appear in Appendix 12. The form to be completed by the teachers who were with their classes when they were tested appears as Appendix 13.

The schools asked to do the Ratio paper in 1977 were:

2nd year. Five comprehensives and two middle schools.

3rd year. Three comprehensives and two grammar schools.

4th year. Five comprehensives.

Some details of the school organisation appear in Appendix 14.

Since the test papers on Algebra and Ratio were the shortest in length, these were sent out as a pair and every child who did Ratio in 1977 also did Algebra. In order to obtain cross matching on as many tests as possible, some schools were asked to give half a year group the Algebra and Ratio papers while others were asked to allow a quarter of the year group to do them. The number expected from each school and the use of the Kolmogorov Smirnov technique to compare the sample with the normal distribution on IQ appears in Appendix 15.

The number of children who actually took the Ratio paper in 1977 was 743. In addition the children who were part of the longitudinal study were also tested but their results were not included in the overall analysis unless those same children were a subset of the school group already being tested.

The crossmatching test against test was composed of fractions of year groups as follows:

Table 7. 1977 Ratio Sample Matched with Other Tests Completed.

	Algebra	Rotation Reflection	Measurement	Fractions	Decimals
2nd yr.	$\frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$		$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}$
3rd yr.	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}$
4th yr.	$\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{4}$

(Each fraction represents the fraction of a year group in one school)

Table 8. Number of Children Attempting Two Test Papers (1977)

	Algebra	Rotation Reflection	Measurement	Fractions (1,2)	Decimals	Fractions (3,4)
RATIO	718	116	138	51	184	68

The Longitudinal Study

Once a hierarchy of understanding had been established, each child was to be assigned a level of achievement (according to the highest group of items on which he obtained a 2/3 pass mark). It was thought likely that the percentage of children at specific levels in each year group would vary. If one was to say that a child in the fourth year was more likely to be in the highest group than a child in the second year and that there was a progression from group to group comensurate with age, then evidence obtained from the same children would complement this. In addition a study of the performance of the same children from year to year would give evidence of how many regressed from one level to a lower. A small number of regressions would demonstrate that the levels were based not on memory alone or briefly learned skills but on something closer to understanding. A longitudinal study was therefore thought necessary, the sample being taken from the second years in 1976 since the IQ of this group was known and they would still be in school after a further two years. The Ratio test paper was given to these children at yearly intervals in 1976, 1977 and 1978.

Sample for the Longitudinal Survey

The sample for the longitudinal survey had to be taken from those children who had been given the Ratio test in their second year in 1976 and whose IQ was known. This limited the schools to four as it was felt that since an equal number of children from each IQ group was needed (as far as possible) then the grammar school which had no children in the lowest IQ groups should be discarded. The plan therefore was to ask that certain children in each school be again tested on Ratio in the summer of 1977, and 1978. The choice of children in any IQ group was made by using random numbers.

The original plan was as follows:

Table 9: The Sample Chosen for the Longitudinal Study

School IQ Range	$89 \leq IQ$	$90 \leq IQ \leq 99$	$100 \leq IQ \leq 109$	$IQ \geq 110$	Total in yr.
A	16	15	16	15	62
B	11	9	11	8	39
C	7	9	4	10	30
D	17	16	20	15	68

The burden placed on the schools in 1977 was considerable in that the children chosen were not in any particular class and they would have to be withdrawn from normal lessons in order to be tested. In 1977 one school refused to do this and offerred to test a low ability set and a high ability set in their now third year, this meant that for this school the children chosen for the longitudinal study were replaced by those who had been tested, in some IQ bands considerably reducing the number of children. A third testing took place in the summer of 1978, by now the number of children who had been absent at one or more of the testing sessions had risen. Those for whom results are reported in a later chapter completed all three tests and were ninety-nine in number. The schools and IQ bands from which they came are shown in Table 10 below:

Table 10 Number of Children Completing the Ratio Test Three Times.

School IQ Range	$89 \leq IQ$	$90 \leq IQ \leq 99$	$100 \leq IQ \leq 109$	$IQ \geq 110$	Total in yr.
A	4	8	6	8	26
B	7	6	6	5	24
C	2	6	3	8	19
D	8	8	8	6	30
Total	21	28	23	27	

The Problem of Grouping Items

Prior to the analysis of the Ratio paper in 1976 the only test analysed by the mathematics team of the C.S.M.S. Project was that of 'Number Operations'. The team had committed itself to writing and administering thirteen tests; a method of grouping items which could be applied to any one of these was therefore very necessary. The grouping of items was needed for two reasons, firstly so that types of questions in any one topic could be ordered according to difficulty and secondly so that children could be ordered according to the types of questions they appeared capable of solving. The child would be assigned a level or declared to be at a stage of understanding in the topic under consideration. The concept of stage as discussed by Wohlwill appeared in chapter two. The stages sought in the present analysis were more analogous to those described in Piaget's work than in "the sequence of skills" type of research carried out by Gagne et al.

If we assume a test has been given to a well-chosen group of children, in the sense that teacher-effect has been spread by using input from more than five schools in different geographical locations and that the whole range of IQ is present, we have initially the facilities for the items from the whole sample and probably from different age groups within the sample. Information for teachers would be best given in terms of types of items rather than individual items. Grouping items together within facility bands gives us a picture of the hard-easy spectrum but provides us with limited information. The easiest items are obviously successfully completed by most children, the hardest by few, those few tending to be brightest. The items having facilities between say 50 percent and 20 percent could be done by different sets of children. If one takes as a premise that items which one groups together must have been done by the same children then one is looking rather more closely at performance than at facility. There are two forms of grouping which immediately arise in such an analysis:

(1) Grouping of items within a limited facility range (horizontal)

(2) Grouping items to form a chain of difficulty easy-hard (vertical).

Fig.4 Grouping Items

	0	0	0		0
Facility	0	0	0		0
	0	0		0	
		0		0	
	0	0		0	

Note: Three chains or vertical strings (one with two branches initially) and three or four groups within facility bands horizontally.

Techniques for finding vertical strings are numerous, for the horizontal grouping there are few, although such grouping would give more information than a chain of single items where one is reduced to talking about an item rather than a type of item. On the basis that one is interested in the items which are done by the same children one must investigate the four cell matrix matching two items.

Fig 5. Four Cell Pass/Fail Matrix

		Item 1 Fail	Item 1 Pass
Item 2	Pass	a	b
	Fail	c	d

a, b, c, d number of children.

If two items were approximately the same facility. one would want both cells d and a to be empty. If item 1 was harder than item 2 one would want cell d to be empty.

Both Loevinger and Yule Q homogeneity coefficients use this cell information, having two items perfectly homogeneous when the coefficient equals 1 and heterogeneous when the coefficient equals 0.

The Loevinger coefficient is $H = \frac{bc - ad}{(b+d)(c+d)}$ (1947)

$\text{Yule } Q = \frac{bc - ad}{bc + ad}$ (1912)

More has been written about Loevinger and in addition it appears to be more difficult to get a high H than a high Q (e.g. using both on the same data). Unfortunately although we have H = 1 as perfect and H = 0 as the worst case, we have no information on the relative adequacy of an H less than 1. In a consideration of H one finds that the facilities of the two items being considered are crucial. For example it is relatively simple to obtain a high value of H when comparing items of 90 percent and 12 percent; one is dealing here with a small number of children, they are successful at the most difficult item and so would rarely fail at the easiest.

Various researchers have used measures of association to form groups of chains of items, the work of some which appeared relevant to the problem presented in the analysis of the Ratio test appear below.

Linkage Analysis (McQuitty 1957)

McQuitty illustrated his method of linkage by using data obtained from the responses of fifteen people to 121 statements on introversion and extraversion (Stephenson's data). His typing is that of people but he started by using the correlation matrix person/person. Applying this method to the analysis of items on a test, one would look at the item/item association matrix, using a homogeneity coefficient for the measure of association. Firstly one underlines the highest value of the coefficient for each item, then one chooses the highest value appearing in the entire matrix. The two items possessing this highest value form the first pair of the first type. Next one connects to these two items every item which has its highest coefficient with either one of them. The process is then continued using items whose highest coefficient is with the appendages. When type one is exhausted one finds the highest coefficient still not used and the process is continued with type two items.

Bart and Krus (1973) proposed another analysis, again involved with a vertical chain rather than horizontal grouping. They investigated two items i and j, if (0, 1) is (incorrect, correct) their interest was in the cell of the pass/fail matrix which displayed item i - 0, item j - 1; item i is called a pre-requisite for item j if a zero appears in the cell which shows fail item i, pass item j. Zero of course appears seldom when real data are used and one has to state a tolerance level. The drawback of dictating a tolerance level is that a low level such as 1 percent picks up little other than that the easiest are prerequisite for the hardest, a higher tolerance level such as 5 percent makes it difficult to distinguish between the easiest items. In order to obtain a tree of prerequisites one needs a higher level such as 5 percent but to distinguish between the easiest, one needs a lower level.

McCready and Merwin (1973) investigated what they called "Item Forms", this consisted of an "item form shell" which provided the general framework within which the item content was to be presented and the "replacement-set structure" which was a rule for generating the content to be used with the item form shell. In particular the authors were investigating relationships among items within item forms and pursuing the notion that if a person gets one question in an item form correct he will get all items within that form correct. An item form is considered inadequate for use in a diagnostic domain referenced test if a) the items within the form are not homogeneous; b) the items are not of equivalent difficulty or c) both of these.

To study the nature of the relationships among items within item forms, they used Loevinger's index of homogeneity H_t (Loevinger 1947).

$$H_t = \frac{\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P_j \, (P_{i/j} - P_i)}{\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} P_j \, (1 - P_j)} \qquad (P_i \geqslant P_j)$$

where P_i is the probability of passing the i^{th} item
P_j is the probability of passing the j^{th} item
$P_{i/j}$ is the probability of passing the i^{th} item among those known to have passed the j^{th} item.

They took $H_t \geqslant .50$ to show relative homogeneity and supported their findings by using McQuitty's Linkage Analysis.

Guttman Scalogram Analysis (1973) was used initially on data obtained from attitude surveys. Guttman defined a scale thus: if one response category is higher than another then all people in the higher category must have higher scale ranks than those in the lower category. In addition each person's response pattern must be deducible from his rank alone. A perfect scale would have all people fitting into the following types (four items ranked in difficulty).

Fig. 6. Guttman's Scale Types

Type Score	Item 4 (Hardest)	Item 3	Item 2	Item 1 (Easiest)
4	1	1	1	1
3	0	1	1	1
2	0	0	1	1
1	0	0	0	1
0	0	0	0	0

0 wrong
1 right

Thus a person with type score four has answered positively exactly the same questions as the person with type score three but has done something extra. There is seldom a perfect scale if real data are used so Guttman suggested a Coefficient of Reproducibility to test how near a perfect scale one is. This coefficient is a measure of the relative degree with which the obtained distribution corresponds to the expected distribution on a perfect scale. Guttman proposed that a coefficient of .90 was acceptable as displaying a perfect scale. This means that if five items are given to a hundred people, resulting in five-hundred responses, one could have fifty errors within that scale. One can however obtain a .90 Coefficient of Reproducibility by chance, so it is suggested that one always check that the number of people in each type obtained by chance is significantly different from that obtained from the data.

A second test of the adequacy of the scale is the Coefficient of Scalability which is $1 - \frac{\text{errors}}{\text{maximum errors}}$. A value of .6 and above is suggested as adequate (Menzel). Further investigations of the errors (people out of type) suggested are:

(1) Errors should be random

(2) Marginals should be balanced

(3) There should be less error than non error.

McCready and Merwin's form of analysis is the only one which investigates the association of items at the same facility levels (horizontal grouping), all the others are essentially testing an association between items, irrespective of facility. None of these methods of analysis seemed to be quite what was needed if one required both horizontal grouping and vertical chaining. An amalgam of these types of analysis was finally adopted, it is discussed in the chapter which reports the results of the survey.

This method involved grouping items of approximately the same facility if the item / item homogeneity coefficient ϕ between any pair was above a stated criterion value. The cut -off between groups was decided on 1) the existence of a facility gap 11) the coherence of the mathematical descriptions that could be given to the items and 111) the methods children used in the interviews. Each child was assigned to a level on the basis of the hardest group of items on which he scored 2/3 correct. The scalability of the groups was tested by using Guttman Scalogram Analysis.

CHAPTER FIVE

Strategies Used by Children in Solving Ratio and Proportion Problems.

This chapter is concerned with the methods children use when solving ratio and proportion problems. An insight into methods available to the children was gained from the interviews with over thirty subjects. Many of the methods used by the children led to specific incorrect answers. These identified incorrect answers were coded when the tests given to a large number of children were marked. The incidence of specific errors, obtained from the wide scale survey is discussed at the end of this chapter.

Interviews with Children

The items which later appeared in the Ratio test were used as an interview instrument with some 35 children. The child was asked to explain how he attempted to solve each of the ratio and proportion problems posed. The changes made in items as a result of difficulties met in the interviews have already been reported in the chapter dealing with the construction of the test. This chapter deals with the information obtained from the interviews which illustrates the methods children use to solve problems in ratio and proportion. An interview lasted approximately one hour so some children did not attempt all the questions. The items were continually modified during the phase of the study in which interviews were held. In the following discussion the final form of the item is given to illustrate the methods used by children but mention is made of earlier versions.

The children interviewed were from a number of schools and of varying abilities. Initially five children from a selective Grammar school were interviewed, the hypothesis being that if these children could not understand the questions then children of less ability would find them very difficult indeed to understand.

Two methods of interview were used, one when the interviewer questioned one child, asking all the time why certain methods had been used and what the child was thinking as he did a question.

The other method (used once with eight children) was to present each child with the test paper and ask him to talk his way through the question, this session took place in a language laboratory with the interviewer "listening-in" to each child's responses. The individual interviews were taperecorded and later transcribed. In a one to one situation the interjections of the interviewer were more numerous and sometimes the child changed the method he was using naturally because a question from the interviewer suggested to him that he was incorrect. In the language laboratory intervention by the interviewer was less frequent but this had the disadvantage that some children gave very little information on how they were doing a question. One child in fact turned off her taperecorder as she said she was 'thinking'.

In the following descriptions of the interviews each question on the final paper will be shown and correct and incorrect strategies described for that particular question.

Question 1 The recipe question

Onion Soup Recipe for 8 persons

8 onions
2 pints water
4 chicken soup cubes
2 dessertspoons butter
$\frac{1}{2}$ pint cream

a) I am cooking onion soup for 4 people.
How much water do I need? ....
How many chicken soup cubes do I need? ....

b) I am cooking onion soup for 6 people.
How much water do I need? ....
How many chicken soup cubes do I need? ....
How much cream do I need? ....

Part (a) essentially required halving, all the children on interview (seventeen in number) used this method, none tried to find the amount for one person in order to use it as a rate, this would of course entail the manipulation of fractions. Part (b) produced a wider selection of methods especially the last part (cream) where often the method which worked on whole numbers was abandoned because it involved the manipulation of fractions. The most popular method for finding the amount of water and soup cubes for six people was to build up to an answer by taking half of a half, as in this reply:

> "Its a pint of water for four people, half of that for two people, add them its 1½ pints."

A lower level of reasoning was to say that six came between four and eight so the amount for six people should come between those for six and eight people:

Child	"4 for eight and 2 for four, it is three, that is halfway between 2 and 4".
Interviewer	"2½ is between as well, why 3?"
Child	"No, I mean equally"

Finding a rate per person was used by a few children on part (b) as in:

> "8 people, a quarter of a pint of water was allotted to one person, its six quarters."
>
> or another child "½ pint for two people, multiply it by three."

Only two children of the seventeen interviewed on this question immediately chose to multiply by ¾:

> "Its six instead of eight, thats threequarters of everything."

When faced with the amount of cream for six people very often the method was changed, the child who multiplied by threequarters said "For cream I did it differently, for eight people you need ½ pint so that is 1/16 for every person. So for six people that is 6/16 or 3/8." The other child who had multiplied by three-quarters reduced the half pint to gills and then found threequarters.

Many of the children who had taken half of a half continued to do so but could not handle the addition of two fractions:

> "Four people a quarter, its 1/4 and 1/8, 1/8, 1/8 is half of 1/4."

Interviewer	"How much is 1/4 and 1/8?
Child	"Is there another name for both of it?" "It was ½ pint for eight people, half of that is 1/4 pint for four people, six is in between, its halfway".

Others immediately said the answer was a third because that fraction was between one quarter and one half. Some children who had been using the half of a half method also changed when dealing with the cream question:

> "A half is four eighths, one eighth of a pint for two people, three eighths for six."

Others continued to use the half of a half and successfully dealt with the addition of the two fractions. The cream part of the question caused considerable unease however, when asked by the interviewer why this part was more difficult one child who had been successful on all the other parts of the question said:

> "Its the fractions. The answer is 1/3, its going to have to be more than 1/4 and less than 1/2."

In the light of what the children did, question one essentially required the ability to halve, this procedure being used twice when the amounts for six people were required. The specific error investigated on the large scale testing was the answer "one third" for the cream question

Question 2

> In an office Mr. Adams comes in to work 2 days a week.
> Mr. Brown comes in to work 4 days a week.
> Mr. Carter comes in 6 days a week.
> The bill for lighting the office for these three men is 240p.
> How much should each pay for it to be fair?
> Mr. Adams .... Mr. Brown .... Mr. Carter........

The most popular method of solution was to find the total number of days involved - twelve and then divide twelve into 240, the resultant of twenty was then multiplied to find the amount payable by each person:

> "the average cost we find by saying 12 into 240, that is 20p. 20p is the average for each man if he came in each day. So A pays 40, B pays 80 and C pays 120."

One child(mentioned above) who multiplied by a fraction in the cream question used a similar procedure in this item (the version interviewed had 'cost of coffee' instead of 'cost of lighting'):

> "Mr. Adams should pay 1/6 of 240p, thats 40p. There's 12 altogether. The number of days they come to work. If Mr. A comes into work two days of the 12 days that coffee is drunk, that is 2/12 or 1/6 of the total cost. Mr. Brown would pay double, Mr. C triple."

Some children recognised that Mr. C came into work the same number of days as the other two men together so reasoned that he should pay half of the 240p, the distribution of the remainder was often a case of trial and error. When pushed some children produced reasons for their answers which were obtained after removing the original 120, e.g.:

> "Mr. Carter should pay half, because those two together make six and he has six. 120 pence and then another 120p left over, which is 80p for Mr. Brown and 40p for Mr. Adams."

Interviewer	"Why 40 and 80?"
Child	"I thought if that would be three days it would be 60, half of it, but its one more. 80 and 40 add up to 120."
Interviewer	"So do 50 and 70."
Child	"He comes in 4 days a week, he should pay double his- because he comes in 2 days."

When the question involved coffee some children were convinced that one of the men must pay a pound even when they had obtained the number eighty by division:

Alicen aged 11	"Is that £240 for a week?"
Interviewer	"It doesn't say, perhaps they settle up the bill every now and then."
A	"Oh do we divide by 3?"
I	"Try it and see."
A	"That's 80p each."
I	"Is that fair?"
A	"No, Mr. C should pay more....then its 60, 80, 100."
I	"Does that add up?"
A	"Yes, its fair now."

Brian concentrated on the difference between the number of days worked and tried to keep a pound as the amount paid by one of the men:

Brian	"Would it come out equally?"
Interviewer	"The men don't go to work equally though, do they?"
Brian	"80, 60, 40. I'll add it up. No £1, 80p, 80p."
Interviewer	"Why?"
Brian	"Because there's two days in between each so that would be 20p in between."
Interviewer	"Why not 25p?"
Brian	"Carter has to pay most. I thought he'd have a pound. Then what do you add to a pound to make 240p. They don't work the same amount. I started with a pound. Each sixty, then eighty."

Lastly a few children took no notice of the number of days each person was in the office and simply divided 240 by three.

Question 3a

There are 3 eels A, B and C in the tank at the Zoo.

15 cm long A

10 cm long B

5 cm long C

The eels are fed sprats, the number depending on their length. If C is fed two sprats, how many sprats should B and A be fed to match?

Part B........ A........

(1) If B eats 12 sprats, how many sprats should A be fed to match?

Part (2) A.........

If A gets 9 sprats, how many sprats should B get to match?

Part (3) B........

The first eel question had been modified from Piaget's original. The first part gave the amount of sprats for the smallest eel which probably provided a clue for the method used by many of the children in the subsequent parts of the question. Part (1) was almost invariably solved by doubling and trebling the amount given to the smallest eel e.g.

"C is two, times it by 2 and then times it by three."

Others used a slightly modified form of this reasoning, talking more in terms of a rate for five centimetres of eel:

"Its 2, 4, 6.. Five, ten, fifteen goes up in fives. Its five up to there, add another 5 which is two, add another five - its another two, so 2, 4, 6."

David provided a mixture of the two methods:

"He's twice as big so 4 sprats. He's (A) twice as much and another five so six sprats. The five left over is another two sprats."

In the second part we are given the number of sprats for the ten centimetre eel so trebling cannot be used immediately neither can a rate since it has first to be found. A sophisticated method from David (who had been given the version of the paper in which B gets 4 sprats) was:

"The answer is six, if he's another half of his size. Like another 1/3 really, let's imagine he (B) is 2/3, so add another 1/3 which would make it six".

Interviewer	"Why thirds?"
David	"Well they're both multiples of 5 and 5 goes three times into 15. So just imagine they are thirds, 1/3 would be half of his size (B) so that is two, so 3/3 would be six."

The finding of a rate is illustrated by the following, both on the 'B has four' version of the items:

Child A	"C first, half of B gives C. A gets three times as much as C."
Child B	"Four for every ten, A has one ten and half of ten."

James seemed to be using the same method but he explained finally in terms of a number pattern:

"Half of four and then add 2 more for that one."

Interviewer	"Why did you add two?"
James	"Because you'd taken two for that one below."
Interviewer	"Why did you not choose three?"
James	"Don't know, it goes up in twos, equal number."

The number pattern reply occurred less often when B was given twelve sprats. Piaget had identified the strategy of just adding one for larger amounts, two children on interview did this; he also identified a later stage (stage III) where the child added a fixed amount for each increase in eel length, the amount being constant but incorrect. An increase of both two and one i.e. the answers 13 and 14 for the amount of sprats for A, were investigated in the large sample testing. The answer obtained by simply doubling (24) was also coded as was a parallel to the addition strategy (identified by Karplus) where the child reasoned the number of sprats was two more than the number of units of length of the eel. This particular error did not appear on this part of the question on interview.

On part three the amount fed to the largest eel was given, the amount for the middle sized eel was required. The child who increased by one for larger sprats now decreased by one for smaller sprats. Gary who had been successful in the previous two questions now gave up: "I think it could be 9 minus something." Dawn who also had the first two parts correct now halved the nine then halved again, finally obtaining $4\frac{1}{2} + 2\frac{1}{4}$ for B. The problem was decidedly more difficult than the two already posed.

The most popular correct method was to divide nine by three to obtain the amount for the smallest eel and then to double to find the amount for B. This method degenerated into a number pattern reason for some of the children: "It just goes down in steps of three." David who had been using thirds already was the only child on interview to mention two thirds explicitly:

> "Each third would have 3 and he only has two thirds so that is six."

Question 3b

Three other eels, X, Y and Z are fed with fish fingers, the length of the fishfinger depending on the length of the eel.

25 cm long Z

15 cm long Y

10 cm long X

Part (1) If X has a fishfinger 2cm long, how long should the fishfinger given to Z be? Z..........

Part (2) If Y has a fishfinger 9cm long, how long should the fishfinger given to Z be? Z..........

Part (3) If Z has a fishfinger 10cm long, how long should the fishfingers given to X and Y be? X........ Y..........

The second eel question had also been adapted from the Piaget version so that the eel lengths were not integer multiples of the smallest. There was a greater variety of methods. Two children multiplied by $2\frac{1}{2}$ to find the length of fishfinger for a 25cm eel given the amount for the 10cm eel but most who had the correct answer for this part of the question built up to the answer in a variety of ways. Some used the fact that the length of X added to the length of Y was the length of Z:

Child A	"10 gets 2cm. 15 is half again. The amount 15 gets plus the amount that 10 gets is the amount 25 gets."
Child B	"Y is $1\frac{1}{2}$ times X, Z is X and Y together."

Others reasoned that 25 was a ten another ten and a half of ten giving:

"The answer is 5, two of those (X) would make 20 and then another 5 which is 1cm fishfinger."

Tony reasoned similarly but became confused over exactly what he was doubling:

"Y is $1\frac{1}{2}$ (times 10), Z is $2\frac{1}{2}$, you double it and half again, the answer is $2\frac{1}{2}$."

Finding a rate for five centimetres of eel was not as popular as previously:

"2cm to 10cm, then its 1cm to 5cm eel. Five fives are twentyfive."

The correct answer was also found from a type of number pattern very close to a rate argument:

"5 into 10 goes 2 and X has 2;5 into 15 goes 3 so Y had 3; 5 into 25 goes 5, Z has 5."

Incorrect methods were numerous, adding on one to give two, three, four occurred: guesses and doubling were also in evidence:

James	"Its 5 and 8. It just goes up more and more.
Interviewer	"Why did you decide 2, 5, 8?"
James	"Well they're 10,15, 25 doesn't go to 20 So guess it goes up more."
Interviewer	"Why 5 and not 6 for Y?"
James	"Suppose it could do but 5 goes into that and that. Then you get another number for that one, more. You could have had 10."
Interviewer	"You could have had ten but you chose eight."

Hames "They're more equal but either would be right."

Another child: "Its 4 and 8. That is double the two makes 15."

Interviewer "Why did you double Y is not double X's length is it?"
Child "No."
Interviewer "Would doubling make it fair."
Child "Yes."

The second part presented more difficulties, apparent in the floundering that took place as children tried to produce a larger number than nine and a cogent reason for having that particular number. Three children did some form of multiplying by 2/3, nobody multiplied by 5/3 (total interviewed was twentynine). Hugh explained his use of 2/3 by:

"I took away 1/3 of Y to get 6 and added 2/3 to get 15."

Janet "10 is 2/3 of 15. 2/3 of 9 is 6. That is 5 + 10 so add those two amounts."

Two children used a rate, another produced this argument:

"Times it by five, they both go into five. 3 fives there. Three times could be three threes which is nine. Five would be 5 threes - 15. I'm not really sure why."

One child attempted to get a ratio by dividing the length of one eel into the length of another:

"Y has 9cm. Divide 9 into 15, goes once with four left over. 1.4 into X goes 7 which is 9.8. So its 7.2cm long."

Variations on the addition strategy also appeared:

"Nineteen. Take away 9 from 15, you get 6. If you take 6 from 25 you get 19."
Interviewer "Why are you subtracting fishfingers from eels?"
No reply!

Part three again gave the amount for the largest eel and required the length of fishfinger for the smaller eels. The interviewer asked a child:

"Why are these end ones more difficult?"
Child "They're the minus ones."
Interviewer "Why do you say minus?"
Child "That's bigger so you really have to do minus to get there, but you could share."

Most of the interviewing was done with the length of fishfinger being five centimetres and not ten as in the final version. Two children dealt with the problem by using 3/5, others concentrated on the factor of five but often as a number pattern: "5 into 25 goes 5, 5 into 10 goes two".

David arrived at the corredt answer (when Z had 5) by a mixture of methods:

> "I'll do X first its a bit easier. 2½ (X) of these go into that (Z). I'll divide by two first then a half, 25. I don't quite understand how I work it out but I think that this has two and this one three".

David	"If 5 goes into 25 five times, and that is 5cm and 5 into 15 goes three so that should have 3cm. It looks as if its right".
Interviewer	"How do you know"?
David	"Thats (X) a bit smaller than that (Y) and it wouldn't be fair to give him 1, so 2 and 3".

Brian started with the 25 and split it into 10 and 15, the lengths of the other two fish:

Interviewer	"Why not 4, and 1 for the other two fish"?
Brian	"No, then X would be 5cm long and Y would be 20cm long."

Halving to find the amount for Y and then halving again for the amount for X also occurred. Many children did not attempt the question.

The methods used by the children involved addition to a very great extent in that the children seemed to prefer to find segments of the answer and add them together rather than multiply by a fraction.

It was apparent that very few children kept a consistent strategy all the way through the two eel questions. If we label the strategies (F) multiplying by a fraction, (R) using a rate, (B) building up to an answer, (A) addition strategy, (P) Piaget level of adding or subtracting 1, (D) doubling or halving when inappropriate, (RB) finding a rate and then building up instead of multiplying, (N) number pattern, (RN) number pattern closely akin to rate finding, (J) adjusting to "make it fair", (FB) fraction and building up; we find the children interviewed performed as follows:

Table 11. Methods Used on the Eel Question

	Eel Question 3a 1	2	3	Eel Question 3b 1	2	3
James	R	RB	RB	N	N	D
Mark	RN	N	RN	D	N	J
Patrick	R	N	R	R	F?	RN
Kim	R	F	R	R	R	R
Paul	R	R	R?	RB	RN	RA
Gillian	R	R	R	R	R	R
Albert	RB	N	P	P	D	J
Brian	RB	RB	R	RB	R	R
Brian (2)	R	RB	RB	RB	D	RB
Mark (2)	R	?	R	F	F	RN
David	RB	FB	F	RA	F	RN
Dawn	R	?	D	correct	correct	RN
Hilary	RN	N	N	RN	omit	
Janet	R	R	R	RB	FB	FB
Hugh	correct	correct	RN	R	FB	
Tim	R	FB	FB	FB	FB	FB
Stephen	N	FB	N	N	N	omit
Lee	correct	RN	RN	N	omit	omit
Colin	correct	correct	correct	A	guess	guess
Fiona	P	P	P	-	-	omit
Louise	correct	P	P	correct	P	-
Alicen	P	P	P	R with help	R with help	R with help
Rosemary	R	F	FB	R	-	-
Gary	RN	RB	omit	F	RN	omit
Tony	R	R	R	A	A	A
Peter	R	R	RB	F	A	omit
Sally	R	R	correct	RN		R
Richard	R	RB	A	P	A	omit

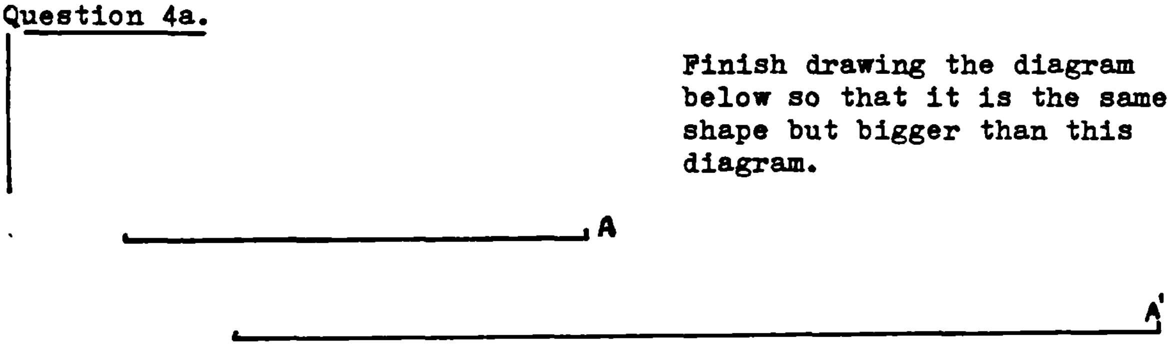
Note: In some cases the method was not available from the tape recording.

Codes: (F) fraction multiplier, (R) rate. (B) building up. (A) addition strategy. (P) adding or subtracting one (D) doubling or halving. (RB) rate then building up. (N) number pattern. (RN) number pattern akin to rate. (J) making it fair. (FB) fraction then building up.

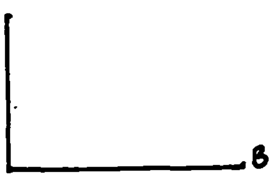
The assignment of the children to categories has of necessity meant that slight differences in the use of a particular strategy have been put together - particularly in the "addition strategy". The interviews were conducted with different versions of the amounts given to each eel, some of which may have pointed the child towards a particular strategy e.g. Alicen obtained 1, 2, 3, for 3a part 1 which was correct and then immediately gave 3, 4, 5 for the second part. It is however apparent that one consistent method throughout was used by only one child - Gillian who used a rate. Tim used fractions and built up to an answer, 3a part 1 did not require fractions so his method is consistent. The absence of children who consistently solved the problems by multiplying by a fraction leads one to consider these items as needing a rate or a building up strategy - the "building-up" however is much more difficult in 3b than in the recipe question.

Question 4a.

Finish drawing the diagram below so that it is the same shape but bigger than this diagram.



Question 4b



Work out how long this line should be if the diagram is to be the same shape but bigger than the one above.

Proportionality in the case of open figures is mentioned by Piaget in "The Child's Conception of Space" (p.372). He used for interview a horizontal straight line (a) six centimetres in length, and another (b) three centimetres in length and drawn perpendicular to but 1.5cm from the end of the horizontal line thus: . Piaget stated that the stages in the discovery of proportion of the open figure appeared to follow the same order as that of the rectangle without being identical in every respect. It is at the age of seven, towards the beginning of substage IIIA that the child makes an advance towards the comparison of lengths, this is the stage in which a) and b) are exaggerated but c) is unchanged. At substage IIIB the child can handle the ratio 1:2 and applies the doubling to length c). The addition strategy is also apparent at stage IIIB, it is not until stage IV (age 11) that metric proportion can be applied to all ratios. The enlargement example used by Piaget was in the ratio 5:3 ; the C.S.M.S. version had example 4a in the ratio 1:2 but containing the gaps as in the Piagetian example just described and 4b in the ratio 5:3 but with no gaps.

There was very little difficulty expressed by the children when asked to double and treble the length of the upright line, although for trebling the method used was sometimes that of adding the original amount to itself twice. The gaps presented some problems, some children ignored them completely providing a resultant enlargement which looked different to the original, others provided a gap without doing any measurement and yet others preserved the length of the gap in the original diagram. The most successful (in the diagram with two gaps) provided a point of reference i.e. the point where the two lines would have met had they been continued. Only one child on interview attempted to use a centre of enlargement, the effort required on the simple doubling question discouraged him from attempting any of the other parts. The item requiring the ratio 5:3 was very much more difficult, the child had first to find how much larger the new diagram had to be and then devise a method. An early version of the question asked for a doubling when the base line was given as 6cm, it then went on to provide a new base line of 10cm and asked for the correct enlargement of the same figure. In this situation some children insisted that the new base line should be 12cm and not 10cm, in fact repeating the question that had gone before. By far the most popular incorrect strategy on the 5:3 item was addition.

> Mark "I'll measure that one, measure that one (given), see how many more centimetres." "That's gone up by four (bases 6 and 10cm). I'll measure the gap, it will be 5½ (1½ in original). Measure the upright, add four onto it, Yes its a scale of four."

In the final version of the paper, (4b above) the answer 'four' could have been obtained by either doubling or by the addition strategy.

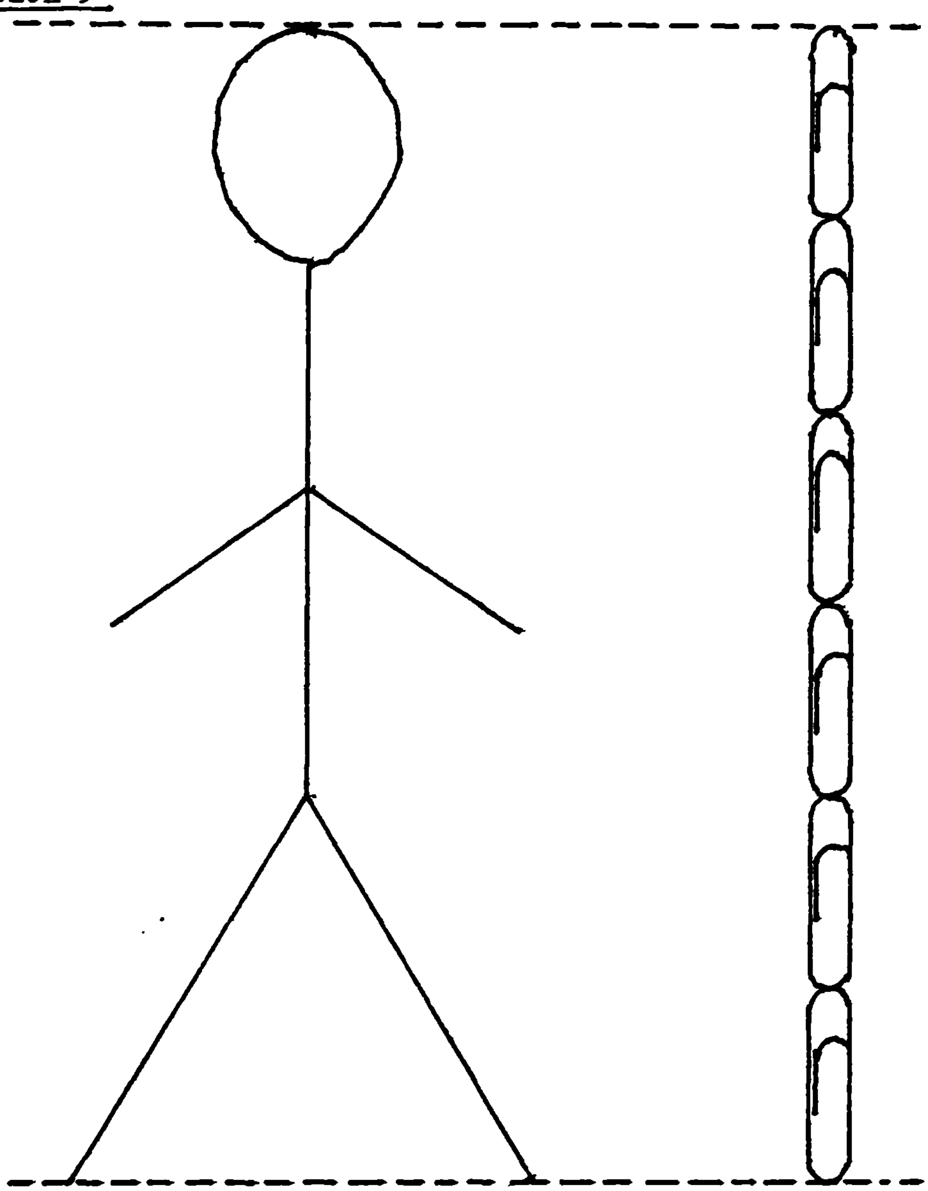
The comparison of the two base lines and then a multiplication by a fraction was used on interview by only one child who in fact made a mistake on the last multiplication:

> " The difference is 4, its like the matchsticks and paper clips. It's 1 2/3 size of that, I want 2/3 of 15 (the measurement of the gap is 15mm). It's ten so I'll have a gap of 2½. 2/3 of six is four for the upright."

Three of the girls from the Grammar school looked at the original diagram and found a relationship between the lines and the gap and then used the same relationship in order to find the new lengths:

Kim	"That would be five, because 6 to 10 so 3 is half of 6, so that would be half of 10, which is 5. 1½ is a quarter of 6 so a quarter of 10 which is 2½."
Child B I	Used a ruler. "Oh, that's 6 and this is 10" Why don't you measure the other pieces in original diagram?
	Long pauses
Child	"3 is half of 6, so must be 5. 1½ is half of 3 so 2½."

Question 5



You can see the height of Mr. Short measured with
paper clips.
Mr. Short has a friend Mr. Tall. When we measure their
heights with matchsticks:
Mr. Short's height is four matchsticks
Mr. Tall's height is six matchsticks
How many paper clips are needed for Mr. Tall's height?..........

The Karplus question used on the 1976 survey was the same size as the diagram above, the 1977 version was somewhat smaller. Correct methods tended to include some use of a fractional multiplier although often in a tentative fashion:

Child A	"Each paperclip is $1\frac{1}{2}$ matchsticks."
Interviewer	"Why?"
A	"6 paperclips and 4 matchsticks. 4 into 6 goes $1\frac{1}{2}$." "9"
Interviewer	"How did you gat that?"
A	"2 matchsticks bigger. (long pause) Its 12".
Interviewer	"Why?"
A	"If each paperclip is $1\frac{1}{2}$ long, its 4 matchsticks. Half is 3 paperclips." Pause.
A	"9. Just add 6 one and halves, makes 9."
Child B	"9".
	"If Mr. Short is 6 paper clips and 4 matchsticks tall, it means paper clip is 2/3 matchstick. 6 matchsticks is 6 times 2/3."
	"You multiply that 6 by 3/2, you turn it upside down".
Interviewer	"Why?"
B	"It's easier to work it out".
Interviewer	"How many paperclips does 1 matchstick equal?"
B	"1 1/3."
	"Same as 4/3, so its 8".
	" It's $1\frac{1}{2}$ paperclips to a matchstick. It's 4 matchsticks and 6 paperclips. 6/4, that's the proportion, it's 3/2. It's $6 \times \frac{3}{2}$, which is $\frac{18}{2}$ or 9".
Tim	"That's $1\frac{1}{2}$. So it will be $7\frac{1}{2}$. Mr. Short is 4 matchsticks and 6 paperclips. For every 4 matchsticks you add 2 paperclips. 4 and half of 4. You got six and half of six - 9."
Interviewer	"Where did the $1\frac{1}{2}$ at the beginning come from?"
Tim	"6 is $1\frac{1}{2}$ times 4."
Interviewer	"Why did you give up that method?"
Tim	"It was confusing - that's where I got that $7\frac{1}{2}$."

Other children started by using a fraction and then abandoned it, illustrated by the following:

Child C	"4 matchsticks, 6 paper clips. 2/3. Divide your 6 by 2/3. No multiply. 4 is 2/3 of 6. So it's, no that's the wrong method."
Interviewer	"What was the 2/3?"
C	"4 over 6". "You add half of that, half of 4 makes 6, add half of that one makes 9".
	"Matchstick 2/3, paper clip 3/3".
	"Matchstick is 1 add a half which is equal to a paper clip".
	"1 1/4 paper clips equal a matchstick".
	"4 paper clips and 2 left over so that's a half. $1\frac{1}{2}$ paper clips equal 1 matchstick".

		P	M
Interviewer	writes as Tall	?	6
	Short	6	4

Child D	"8". "4 to 6 and 6 to 8".
Interviewer	"Any other ideas?"
D	"4 matchsticks doesn't go evenly into paper clips". "Are we allowed to measure these?" (Measures with ruler) "Be about 7½".
Interviewer	"Why?"
D	"It's almost that. I took the height and divided it by 4 because there's four matchsticks".
Interviewer	"Why did you need a ruler?"
D	"4 won't go into 6". "I wanted to see how many matchsticks equalled paper clips".
Interviewer	"4 matchsticks equals?" "1 matchstick equals?"
D	"1½". "Answer is 9".
Interviewer	"Which answer do you want to choose?"
D	"I'll stick to 9". The amount of matchsticks doesn't go evenly into the amount of paperclips".

David was more confident although his first answer was wrong:

> "8. I've gone back to thirds. Each third of Mr. Tall is two matchsticks. Mr. Short is only 2/3. Each third is 3 paperclips. Mr. Tall is 9, another third."

The most popular incorrect answer was the answer 'eight' found by the addition strategy:

> "Mr. Short is 6 paperclips but 4 matchsticks makes it two more. As he is 6 matchsticks, it should be two more."

or "6 equals 4 matchsticks. 6 matchsticks equals 8 paperclips. 2 matchsticks higher so take 2 matchsticks away when you measure in paperclips is 6. So it's 6 matchsticks, take 2 away, so 8 paperclips. He's two more matchsticks."

Lee thought Mr. Tall should have double the number and gave 12 as the answer immediately:

> "12, no about 24. It's 12, for every 2 matches you have one paper clip. I don't know if that is true, I just thought."

Interviewer	"Should you check?"
Lee	"If he were double his size then it still would be 12 paperclips."
Interviewer	"Is he double?"
Lee	"No he's only two matches taller. Matches 4 and 6...paperclips 6 and.....It's about 12."

Karplus had identified the following strategies in the child's solution of the Mr. Short and Mr. Tall question: Intuitive, Addition, Transitional and Ratio. We have described the addition strategy, Intuitive was akin to guessing or not making use of all the data, it would include the doubling strategy mentioned above. The transitional category showed only partial proportional reasoning and would include the reversal of the match/paperclip relationship.

Question 6

In a particular metal alloy there are
1 part mercury to 5 parts copper
3 parts tin to 10 parts copper
8 parts zinc to 15 parts copper
You would need how many parts mercury to how many parts tin?

.... parts mercury to .... parts tin.

You would need how many parts zinc to how many parts tin?

.... parts zinc to .... parts tin.

Few children on interview had a strategy which gave the correct solution, although nobody appeared to have difficulty with the word 'parts'. One child successfully used fractions. She firstly used the chain of reasoning, "1 part mercury to 5 copper, 10 copper to 3 tin, that is 1 mercury to 1½ tin." For the second part she found a rate per 5 parts copper thus:10 copper -- 2 2/3 x 2 = 5 1/3, so 5 1/3 parts zinc to 3 parts tin. Kim tried to make the amount of copper uniform throughout but used 15 rather than 30:

Kim	"You have to make them to get to 15. Multiply that one by 1½. The amounts of that gives you same amount of copper". "4½ tin to 15 copper 3 m to 15 copper It's 1 to 1½. So it's 3m to 4½ tin. 8 parts zinc to 4½ tin" "You can cancel the fraction out". "You can't cancel zinc to tin. You could make them all halves".
Interviewer	"Why did you use 15?"
Kim	"You need the copper to be the same to find the relationship between mercury and tin".
Interviewer	"Could I have used 5?"
Kim	"Yes. You could multiply by ½ and then by 1/3."

Yet others knew that something had to be done to the amounts but were not quite sure what. Part one was more likely to be correct since doubling would provide the answer. Another incorrect method was to multiply both the amounts given, thus satisfying the idea that something had to be done but demonstrating that the understanding of equivalent fractions was rather tenuous. Others doubled one amount and then halved the second amount:

Child C	"2 mercury".
	"If 10 parts of copper that means there'll be 2 parts of mercury".
Interviewer	"2 mercury to how many tin?"
C	"$1\frac{1}{2}$".
	"I don't know why - took 10 and then took the 3 to 5".
	(obviously divided 3 by 2)
Interviewer	"So 2 mercury to $1\frac{1}{2}$ tin?"
(b)	$4\frac{1}{2}$ for tin (had increased 3 by $\frac{1}{2}$)
	6 for mercury (had tried to take 2/3)

One child used 30 parts copper throughout and was both quick and accurate.

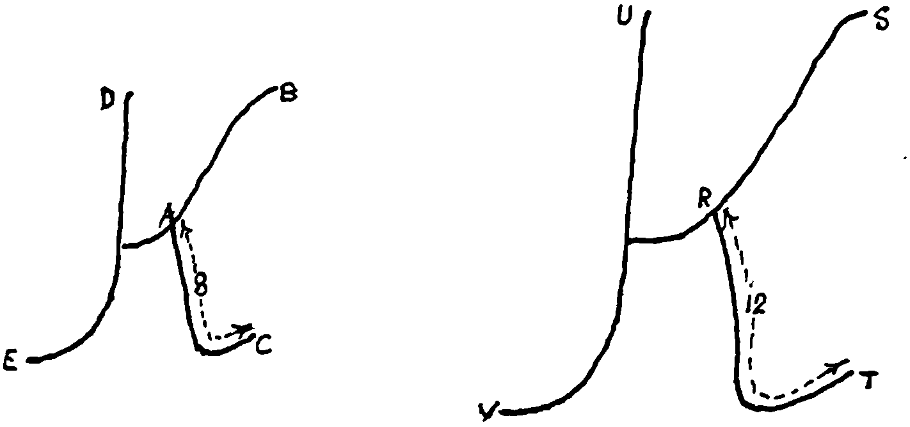
Many of the children however completely ignored the fact that the two comparisons with copper were in fact comparisons with different amounts of copper, they concentrated on the two commodities that were required for an answer:

Interviewer	"What does that word 'part' mean?"
Child A	"It means 1 percent mercury to 5 percent copper".
	"1 part mercury to 3 parts tin".
Interviewer	"Doesn't it make any difference that you have 5 and 10?"
A	"Not really, no".
A	"8 to 3".
Child B	"1 part mercury to 3 parts tin".
Interviewer	"Why did you decide on those?"
B	It's there, 1 part mercury and 3 parts tin".
Interviewer	"Doesn't it make any difference that I have 5 parts copper and 10 parts copper?"
B	"Yes it does I suppose".
Interviewer	"What are you going to do about it?"
B	"Well you can't really work it out like that".
B (ii)	"8 parts zinc to 3 parts tin".
Interviewer	"Does it matter that you have 10 parts copper and 15 parts copper?"
B	"No".

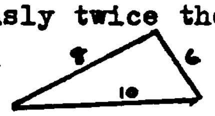
Question 7

These 2 letters are the same shape, one is larger than the other.

AC is 8 units.   RT is 12 units



The curve AB is 9 units.   How long is the curve RS?
The curve UV is 18 units.   How long is the curve DE?

The two figures in question seven were to test for the understanding of similarity without requiring the child to understand the word 'similar' or to draw a diagram. Few children were interviewed using this diagram but it was known from interviews on other similar figures (see pp 74) that the addition strategy was very prevalent unless one figure was obviously twice the size of the other. For example given the triangle  and provided with mechano strips to make a similar triangle the addition strategy showed itself as follows:

Mark aged thirteen used two methods:

M — "Six goes to twelve, eight to sixteen, ten to twenty."
Interviewer — "How did you decide?"
M — "Doubled each one."
Interviewer — "Say if you started with eight?"
M — "Ten would go to fourteen, six would go to ten."
Interviewer — "Why?"
M — "Adding four on, 6 goes to ten, 10 add 4 is 14."

Colin aged fifteen looks at 6, 8, 10 and decides the crucial issue is that they differ by two:

> "They're going up im twos and when I looked at them I see there is 12, 15 and 9. But I was thinking that (Points to 6) being 12, that (8) being 15 and that (10) being 18 but there aint no 18 there. So I put it there (points to 8). That's shorter than that, so it would be 9, and that's larger, so it would be 15. They're going up in threes."

As can be seen unless the child was to draw the new triangle and then have some criteria by which he could judge whether it was the same shape as the original, he had no way of knowing whether his suggestions for the lengths of sides were correct. In each case the child had certainly made the sides longer.

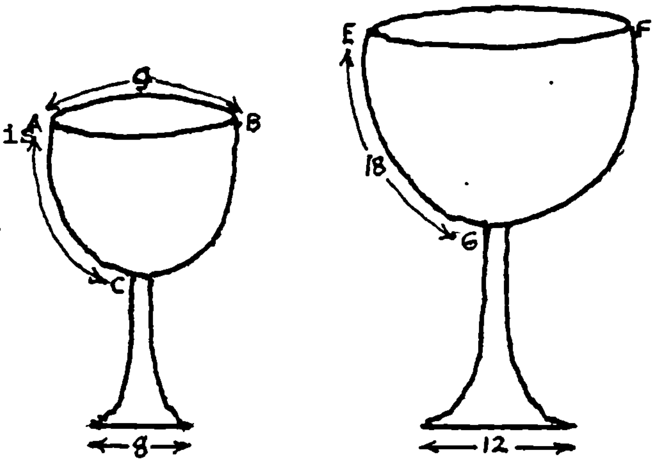
Looking at the letters K, Richard for example reasoned:

"RS is 13. 12 is bigger than 8 by 4. I looked there, AB is 9, one bigger than 8, RS is one bigger than 12, DE is 14, 18 is 6 bigger than 12 so DE is 6 bigger than 8."

Some children were given both the pictures of the glasses and that of the letters, so that in addition to question 7 above they also had:

AB is 9 units. How long is EF?

EG is 18 units. How long is AC?



Peter used a fraction for obtaining the answer but used it in fact to build up to an answer rather than multiplying just once:

Peter solving both the glasses problem and the letters problem:

<u>Glasses</u>

P	"8 is 2/3 of 12. So that is 1/3 larger than that. 1/3 of 9 is 3 so 9 + 3 is 12."
Interviewer	"Bottom of glasses 8 for small and 12 for large."
P	"12. 1/3 off of 18 is 6 off, that is 12."

<u>Letters</u>

"1/3 larger, 1/3 of 9 is 3 that gives 12."

Two children naturally used the addition method until prompted and asked to think again, one was rather more sure of her reasoning than the other however:

Child A. Glasses

A "EF is 13 units. That's 4 more than that, so I assume it will be 4 more in each place."

Interviewer "Why? Can you see any other relationship between 8 and 12?"

A "8 is 2/3 of 12. Can't see anything else."

Interviewer "Which seems most sensible the difference of 4 or the 2/3?"

A "The 2/3 I think. It's just that's a 1/3 bigger."

Interviewer "Can you work it out?"

A "EF will be $13\frac{1}{2}$. $4\frac{1}{2}$ is 1/3, 9 is 2/3, another 1/3 is $13\frac{1}{2}$. AC is 12."

Interviewer "Why was that easy?"

A "3 into 18 goes 6."

Child B Glasses

B "It isn't two times bigger. 4s go into them, that can't help."

Interviewer "Any relationship between 8 and 12?"

B "4 goes into both of them, twice and 3."

Gives up.

Child C. Letters

C "It's the same."

"I could do it if I knew 8 and 12."

Told this picture is 2/3 size of the bigger one.

C "9 times 2/3."

Interviewer "Why?"

C "If its 2/3 larger."

Interviewer "No that's the larger

Question 8

% means per cent or per 100, so 3% is 3 out of every 100.

a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ....

b) 6% of children in a school have free dinners. there are 250 children in the school.
How many children have free dinner? ....

c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.
What percentage is this? ....

d) The price of a coat is £20, in the sale it is reduced by 5%; how much does it now cost?
....

Percentages were included on the paper since they seemed to be an important part of the topic of ratio, fractions as has been described earlier, were left for a separate paper. The word 'percentage' had to be explained to a lot of children although one insisted he had "done percentages in the primary school". The numbers given were all halves of hundreds or an integer number of hundreds, the children who used this fact and built up to an answer were on the whole the most successful, those who tried to remember a technique often misremembered it and proceeded to carry out needless and erroneous computation. Part (a) was a simple repetition of what was on the previous line to explain the meaning of percentage. The building up method is illustrated by the following from Patrick:

> Part (b) "6 per hundred, 2 hundreds is 12 and 50 is half of a hundred so half of 6 is 3. It's 12 + 3."

Half remembered rules were varied and numerous on part (b):

Child A (b)	Divides 6 into 25. "When I've done it I'll put the 0".
	"No it doesn't work".
Interviewer	re-reads question
A	"5% would be OK".
Interviewer	"What does 6% mean?"
A	"6 out of 100".
	"5% you'd need another per cent".
	Leaves
Child B (a)	"4%"
	"You usually work percentage in a hundred per cent, 4 out of a 100 is just 4, 4%"
B (b)	"14½"
Interviewer	"How many children have free dinners, can you have 14½ children?"
B	"Don't know".
B (c)	"30%. 8 threes in 24, add a nought which is 30".
Interviewer	"Why did you add a nought?"
B	"just guessing".

Stephen divided for the first three parts which actually gave him the correct answer in part c:

Stephen

Part (b)	"50 - 250 divided by 6 would make 50".
Part (c)	"30% divide 8 into 24".

James doubled or halved:

Part (b)	"125 You just halve the number of children".
Part (c)	"It's 48%".

Colin had part (b) correct but for part (c) he multiplied 8 by 24; this answer was investigated on the large sample testing. Gillian managed to deal with fractions of children but she was the only one who used this method:

> "250 divided by 100 its $2\frac{1}{2}$ children. Multiply it by 6. I want to find how many children in 6%. How many children in 1% which is $2\frac{1}{2}$."

Mark divided 6 into 250 seemingly working it out by finding 100 ÷ 6 and then doubling, then taking a half and adding:

Mark's paper

4 / 6)250 — % means percent or per 100, so 3% is 3 out of every 100

16r4 / 6)100 — a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ..4%..

16r4 × 2 = 42r8 — b) 6% of children in a school have free dinners. there are 250 children in the school.
How many have free dinner? ..64..

2)42r8 = 21r2, 64 — c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.
What percentage is this? ..3%..

4.08 / 5)20.40, − 4.08 — d) The price of a coat is £20.40, in the sale it is reduced by 5%
How much does it now cost? ..£16.32..

NOTE: (Part (d) was changed to give a coat costing £20 in the final version)

Note that on part (d) he divided the cost of the coat by five and then subtracted, on the large scale testing this error together with simply subtracting the five were investigated, some children it was felt would also forget to subtract the pound having found the reduction so this was an error for investigation when the large sample was tested.

Summarising the responses of the children to the questions asked them about ratio we find that the methods of solution were numerous. Among correct methods the use of the algorithm $x/a = y/b$ was virtually unheard of. Multiplying by a fraction was avoided and when used tended to be part of a solution rather than the method which would give a solution immediately. Multiplication by an improper fraction although apposite in for example the eel question, never occurred. The most usual method of solution was that of trying to build up to an answer so that one repeated an amount and then took half.The incorrect methods were varied but some were easily identifiable, for example, the addition strategy where the child concentrated on the difference between corresponding lengths rather than the ratio. The doubling strategy where the child doubled to make larger and halved to make smaller also occurred. Other incorrect methods where the child has used an incorrect computation or misremembered a rule also appeared fairly often and when recognisable in the answer given, these were coded on the large sample marking.

The Final Test

Mathematical Demand of Items

The final test appears on page 94. The items were initially described in terms of their mathematical demand, for example the type of multiplier needed and whether whole numbers or fractions were involved. A first description was as follows:

Table 12: Mathematical Demand of Items*

No rate needed	Given Rate 1:x	Given Rate not 1:x	Find Rate 1:x	Find Rate not 1:x	Find rate requiring intermediate step	Operation
1, 2	7				16,17,15,20	Whole nos double or halve
3, 4						As above twice
	8		11,6	9,10,27		Multiply whole nos by integer
		25,26		12,13,14 22,23	19,21	x1½ x2½
					18	x3/5
5						Fraction operation

*(the numbers used are the variable numbers quoted on the test paper at the end of chapter 3).

The test paper that appears at the end of chapter 3 was used for the wide-scale testing. Most questions had several parts; in order to distinguish between them each answer coded in the final marking was given a 'variable' number, these are shown on the test paper and on table 15.

The interviews gave some insight into what the children considered was the mathematical demand of the items. Although a teacher might, for instance, consider that the eel questions required the multiplication by a non integer (variables 12, 13, 14) the children actually built up to an answer rather than attempt to multiply by a fraction. This meant that what the mathematics teacher thought was the demand was not necessarily the case and the items had to be investigated more with regard to the most commonly available strategy shown by the children interviewed. Those items which could be solved by repetition and finding a half had proved to be relatively easy in the pilot study. Items where this method resulted in "take a whole and a bit" and "the bit" was not a half, tended to be considerably harder.

Marking Codes on the Final Test

The errors which had become apparent from the interviews were coded on the large scale survey and any other error which appeared in great number at the start of the marking were also coded. Some types of errors (such as those resulting from the incorrect addition strategy) occurred on more than one item and so the same code was given whenever the answer matched that which was obtained when the particular strategy was used. Other errors were specific to the item and the codes assigned are explained below. Code 1 was always given for the correct answer, Code 0 for an omission and Code 9 for an incorrect answer which had not already been identified as arising from a particular method. The complete Coding Scheme appears as Table 13.

Table 13. Coding Scheme

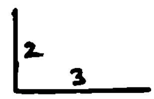
		Codes	Correct 1	2	3	Adding on 1 4	Adding on 2 5	Addition Strategy 6	Doubling 7	Specific error described in "Items" 8	Other wrong 9
Q.1	a	water	1								
		cubes	2								
	b	water	$1\frac{1}{2}$ $(\frac{3}{2})$								
		cubes	3								
		cream	$\frac{3}{8}(\frac{1}{4}+\frac{1}{8})$							1/3	
Q.2	A B C		40,80,120	120,80, 40	Equal steps eg 60,80,100					80,80,80	
Q.3a		B	4			3	-		-	12	
		A	6			4	-		8	17	
		A	18			13	14	17	24		
		B	6			8	7	4	$4\frac{1}{2}$		
Q.3b		Z	5			4	6	17	8		
		Z	15			10	11	19	18		
		X	4			8	6	-	$2\frac{1}{2}$		
		Y	6			9	8	-	5		
Q.4a	Vertical line		4		-			8		-	
	Vertical Gap		2		no gaps			7		1	
	Horizontal Gap		3		shown			$7\frac{1}{2}$		$1\frac{1}{2}$	
Q.4b			3.2-3.5			3		4		2	
Q.5	Mr. Tall		9			7		8	12		
Q.6	Mercury Tin		2, 3 1,$1\frac{1}{2}$		2,6)or other)			1,8		1,3	
	Zinc Tin		16,9 8,$4\frac{1}{2}$		16,6)multiple			8,8 or 3,13		8,3	
Q.7		RS	$13\frac{1}{2}$			10		13	18	-	
		DE	12			17		14	9	27	
Q.8		a	4								
		b	15								
		c	3							192	
		d	19	1						16 or 15	

Three errors which had been identified by Karplus and Piaget and which had appeared on the interviews were coded 4, 5, 6 and are described below. The error of doubling to enlarge (called 'scaling' by Karplus) was coded 7. The specific wrong answers which arose when these particular methods were used appear against the appropriate question in table 13, in the columns headed 4, 5, 6, 7.

a) Adding on 1. The child faced with an enlargement reasons that the resultant must be larger but one unit larger is sufficient; similarly to obtain a smaller amount a subtraction of one unit is sufficient. (Code 4).

b) Adding on 2. This is similar to the above but the child is reasoning that extra units are needed; he has little idea how to provide those extra units. Similarly for providing a smaller amount. (Code 5).

c) Addition Strategy. The child concentrated on the difference a − b rather than the ratio a/b. For example asked to enlarge



so that the new base line is 5 units, he will say 5 is 2 more than 3 so the uprights is 2 more than 2, answer 4 units. (Code 6).

d) Doubling. The child regards all enlargement as requiring doubling and similarly he sees halving as the normal process for making something smaller. The code (7) is used when this is not what is required in the problem.

Methods used by the children when attempting the questions have been described in the section on interviews. It was impossible to ascertain from the written tests which correct methods had been used. The incorrect methods were inferred from the particular incorrect answers. The following discussion summarises the strategies which appeared on interview and describes the codes assigned to specific wrong answers on individual questions.

Question 1.

1a. The results are whole numbers. The answer was most often obtained by halving.

1b. Part a) has a fractional answer. The most common method used by the children interviewed was that of saying "How much for 4 people? How muah for 2 people. Add the two amounts". In part c) considerable difficulty was found since this method resulted in the addition of two fractions. Many children opted for "What is the fraction halfway between 1/4 and 1/2?" but chose 1/3 as fulfilling this condition. (coded 8)

Question 2.

This item requires the child to find a rate of lighting per day and then multiply by 2, 4 and 6. Most children on interview did this or multiplied the rate by 2; then doubled this amount for 4 and then added to obtain the lighting bill for 6. Other children decided that each man should pay the same amount (80p, 80p, 80p) (code 8) yet others required the bills to differ by a fixed amount, the total being 240p, e.g. 60p, 80p, £1 (code 3). Yet others found the correct amounts to be paid but reversed the order (Code 2).

Question 3.

Question 3 is taken from the work of Piaget (Epistemologie et Psychologie de la Fonction, 1968) the numbers used have been changed from those used in Geneva. The most common methods used for solution were to find "how many for 5cm" and then multiply or a variation of 'building up'.

Part 3a) involved discrete amounts and if using a rate for 5cm of eel, only multiplication by 2 and 3 is required; sometimes the child added three times instead of multiplying. We have allowed coding for the addition strategy. This is seen as taking the difference in eel lengths, converting this into sprats and adding to the original number of sprats. Doubling in part 1 (given food for C find it for A) is seen as completing the number pattern 2(C), 4(B), 8(A). There were some children who simply added the length of eel (cm) to the number of sprats, so this

error is coded (8).
Part 3b) The eel lengths are no longer integer multiples of the smallest. The amount of food fed is no longer discrete. The most common correct methods used on interview were to find how much for 5cm of eel and multiply or build up. Adding fishfinger lengths to eel lengths is coded (8). To show doubling the answers coded are those which would arise from consistent doubling (X,Y,Z).

Question 4.

4a. The enlargement is 2:1 but the child must find this by comparing the two base lines. The use of a centre of enlargement is difficult since it comes off the page. The vertical and horizontal gaps should also be enlarged in the ratio 2 to 1; it was found on the trials however that both children and adults forgot this. The important points being investigated however are a) given two line segments can the child find the factor of enlargement, b) can they then double the length of a line either by multiplying by two or by taking the line and then repeating it. The addition strategy would result in the child saying the difference between base lengths is 6cm so the difference between lengths of upright is 6cm, answer 8cm (coded 6) similarly the length of the gaps resulting from addition are coded 6. An omission of a gap is coded 3, leaving the gaps as they were in the original diagram is coded 8.

4b The enlargement of the line segment is in the ratio 5:3. This does not lend itself very readily to the method quoted in item 1 of finding two amounts and adding, (an enlargement of 3:2 could be achieved this way). Usually the children who succeeded attempted $\frac{2 \times 5}{3}$ by some method. The answer 10/3 or any value between 3.2 and 3.5 inclusive was counted as correct. The answer 3cm was also coded, it could arise from adding 1, or from "The 3cm base is contained 'one and a bit' times in the 5cm base, the new upright is 'one and a bit' times 2cm" Code (4).

Question 5.

This item is taken from the work of E & R Karplus (Proportional Reasoning and Control of Variables in Seven Countries). The correct methods of solution are a) using some form of $\frac{6 \times 6}{4}$

b) finding a rate matchstick: paperclip and then multiplying
c) finding the equivalent of 2 matchsticks in paperclips and then using (2+4) matchsticks and their equivalent. The problem requires the child to recognise that ratio is needed, ascertain a correspondence in some form and then find a solution. The addition strategy is coded (6) doubling and adding 1 are also coded.

Question 6.

Question 6 requires the child to use a third commodity in order to find a relationship between two other commodities. Many children of course ignored the first commodity (copper) since it did not appear in the requirements of the answer (coded 8) or doubled the 1 and 3 which were given (coded 3). Correct strategies for mercury : tin were:- a) doubling, b) saying "1 part mercury and another part mercury correspond to 10 parts copper". The zinc, tin ratio could be obtained by multiplying by $1\frac{1}{2}$ or a more popular method was to say "5 parts more copper would give me $1\frac{1}{2}$ parts more tin so $(3+1\frac{1}{2})$ tin to 8 zinc". Some children (few) decided to work with 30 parts copper throughout and adjust the other commodities accordingly.

Question 7. The child must recognise that a ratio is needed by inspecting the two similar figures and then he must compute the factor of enlargement. Incorrect strategies mentioned previously have been coded and in addition on part 2 some children correctly ascertained the ratio 3 : 2 but enlarged the figure UVRST (coded 8).

Question 8.

The symbol % is introduced and each part of the question is solvable by using "parts of a hundred" without recourse to excessive computation. On part c many children tried to remember a rule and used $\frac{800}{24} \times 100 = 192$ (coded 8). Part (d) produced children who forgot to subtract the pound and gave it as an answer (code 2). Others who interpreted 5% as one fifth gave the answer 4 or subtracted 4 from 20 to give 16. Another version was to simply state 5 as the resultant or to subtract 5 from 20. Answers 15 and 16 are coded 8.

Error Incidence on the Final Test

The answers arising from the addition strategy occurred very often on questions 4b, 5 and 7. A large number of children gave 16 or 15 for the answer to 8d. When the incidence of an incorrect code reached more than 10 percent it is quoted below (in table 14).

Table 14. Error Incidence, 1976 Testing. (n=2257)

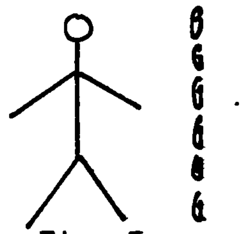
Item		Codes					
		3	4	5	6	7	8
	2nd yr						20.6
1b(iii)	3rd yr						23.6
	4th yr						21.7
2							10.1
							8.2
							5.5
3a(iii)						10.2	
						8.9	
						5.8	
3a(iv)				10.1			
				8.5			
				8.7			
3b(i)				13.4			
				12.1			
				9.7			
3b(ii)						12.5	
						12.5	
						8.3	
3b(iii)						11.7	
						11.6	
						8.8	
3b(iv)						20	
						16.6	
						12.9	
4a V Gap		10.7					19
		9.7					14.6
		6.8					14.3
H Gap							17.9
							12
							10.9
4b			15		47.6		
			16.7		39.4		
			11.7		39.7		
5					51.4		
					50.6		
					39.1		
6a						21.4	
						26.9	
						17.2	
6b						19.2	
						25.8	
						16.2	
7a					43.6		
					40.3		
					34.2		
7b					32.2		
					29.3		
					25.0		
8c							7.4
							10.4
							8.8
							40.9
8d							35.9
							27.4

As can be seen from table fourteen, some incorrect answers which occurred on interview, appeared on less than ten percent of the test papers from each age group. Other errors, particularly those arising from the addition strategy occurred on more than 40 percent of the test papers. The errors which appeared on many papers are now discussed in detail; the implication that certain methods were used to arrive at them is made in the light of the interviews.

The Addition Strategy.

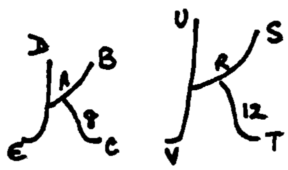
Kapplus has already identified the addition strategy where the child concentrates on the difference between the two amounts in the ratio and views the process as that of addition and not multiplication. Answers arising from the use of this incorrect strategy occurred at a very high level on four of the ratio questions. The questions and the incidence of the addition strategy are shown below (1976 data).

Mr. Short & Mr. Tall



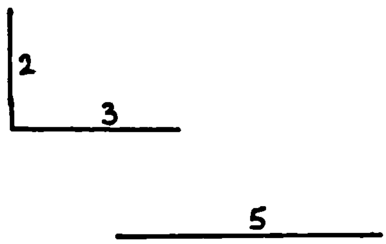
Item 5

	2nd yr	3rd yr	4th yr
Answer 8	51.4	50.6	39.1 per cent
Answer correct	28.1	29.6	42.0



Item 7a, 7b

	2nd yr	3rd yr	4th yr
Answer RS=13	43.6	40.3	34.2
Answer correct	13.7	19.7	28.7
Answer DE=14	32.2.	29.3	25
Answer correct	15.4	20.5	28.8



Item 4b

	2nd yr	3rd yr	4th yr
Answer 4 for upright	47.6	39.4	39.7
Correct answer	7.9	11	19.7

These four questions attracted the addition strategy more than any others. All the items involved a diagram and the answer obtained from the addition strategy was not very different from the correct answer, it was in fact plausible.

Although there were rather more second years using the addition strategy than fourth years, a large percentage of the older children opted for this method on these four questions. Notice too that there was a sizeable difference between the percentage of children using the addition strategy on the two parts of the question dealing with similar figures K. Although the percentage correct was relatively constant, the process required to step down from a larger figure to a smaller seemed not to attract the addition strategy as often as the process required to step up from a smaller to a larger figure.

Since so many children were using the addition strategy on these four questions, the question of how consistently they used the method naturally arose. 622 children obtained "additive" answers on three or four of these four questions, 392 added on three of the four items, the remaining 230 added on all four.

Out of the 622 adders:

152 did not add on item 4b

123 did not add on item 5

103 did not add on item 7b

14 did not add on item 7a

There were some thirty percent of the entire sample using this strategy consistently on three of the four questions which attracted it most. These children were regarded as forming a group which showed some particular error and were called "Adders". Piaget when describing the use of this incorrect strategy (with respect to the enlargement of a rectangle) stated that it was a response symptomatic of the late concrete child. To find whether these children were low attainers on the other ratio items, their performance on each item and later on each level of items was found, the performance appears in table 15 below.

Table 15. The Performance of Adders on Test Items.

Item &	variable no.	Percentage of adders passing	Overall facility	Adding 2	Total Sample n= 2257 Adders n= 622
1a(i)	1	97.7	94.8		
1a(ii)	2	97.6	95.0		
1b(i)	3	90.0	85.4		
1b(ii)	4	78.9	76.0		
1b(iii)	5	17.5	26.0		
2	6	47.1	48.5		
3a(i)	7	87.8	86.4		
3a(ii)	8	76.8	77.8		
3a(iii)	9	41.6	50.2	11.3	
3a(iv)	10	40.5	50.5	13.8	
3b(i)	11	46.1	49.8	16.2	
3b(ii)	12	18.8	30.7	5.9	
3b(iii)	13	19.6	27.2	6.6	
3b(iv)	14	19.9	28.0	8.0	
4a.V.	15	77.5	72.4		
VG	16	41.6	43.8		
HG	17	39.7	42.6		
Questions 4b, 5, 7a, 7b (variables 18,19,22,23) define adders					
6a	20	27.3	32.3		
	21				
8a	24	90.5	85.0		
8b	25	46.0	45.8		
8c	26	36.7	39.0		
8d	27	20.6	27.0		

It can be seen that 'the Adders' are not the least able, they can solve the easiest items e.g. the recipe question, but are likely to add two (Piaget level II) on the eel question, however even on these questions it is only a small percentage of the adders who resort to this strategy. Further information on level of attainment of 'the Adders' is contained in the chapter dealing with the general results of the survey. The adders within the longitudinal survey sample and their subsequent performance is also described in that chapter.

Piaget Levels

Piaget described two behaviours on the eel questions, level 1 - adding just one more for a larger eel and level two - adding a fixed amount (not one) for a larger eel and subtracting the same amount for a smaller eel. The incidence of level one answers was very low, only on variable eleven did it reach three percent. For the purpose of marking, level II was defined as an increase or decrease of two, the incidence of this answer was rather larger:

Table 16: Percentage of Children who Add Two. n = 2257

Variable No	4th yr	3rd yr	2nd yr	
9	7.4	7.2	8.5	percent
10	8.7	8.5	10.1	
11	9.7	12.1	13.4	
12	4.1	4.3	5.5	
13	8.8	5.9	6.6	
14	5.1	6	6.6	

Var. 11 seemed to attract this strategy most often but since the amount given to the smallest eel is two, the result may come about because the child doubles and trebles without seeing that the length of eel is not in the ratio 2:1 or 3:1.

Defining children as being of level two if they add 2 for each increase in length or subtract 2 for a decrease in length we find that although 404/2257 (17.9 percent) use this strategy on one or other of the items 9-14, only two percent use it on four of the six items. The percentage of children who might be said to be consistently at level II is very small.

Doubling for Larger, Halving for Smaller.

Doubling or halving appears to be a relatively simple operation. On interview some children had used this method for solution on items where it was the incorrect method.

The percentage of children who double or halve when this is incorrect is shown in table 17, (the results are from the survey and are based on the frequency of the answer which would result if a doubling or halving strategy had been used).

Table 17: Percentage of Children Who Double or Halve n = 2257

Variable	4th yr	3rd yr	2nd yr	
8	2.0	2.7	4.4	Percent
9	5.8	8.9	10.2	
10	4.2	5.6	7.7	
11	5.9	6.5	5.6	
12	8.3	12.5	12.5	
13	8.8	11.6	11.7	
14	12.9	16.6	20.0	
19	0.9	2.5	2.0	
22	2.6	6.4	6.9	
23	2.0	6.0	8.5	

The eel questions attracted the incorrect doubling strategy most often.

Other Common Errors.

On interview, a large number of children had given the answer 'one third' for the recipe question involving cream ($\frac{1}{2}$ pint for 8 people, how much for 6?). The reasoning had been that the fraction of a pint required for six people was halfway between one half and one quarter and one third satisfied that criterion. In the large sample testing, twenty percent of each year group gave 'a third' as the answer.

The chemical compounds question was difficult and very nearly twenty percent of each year group ignored the amount of copper entirely. They gave the ratios between the two metals exactly as they appeared in the question.

A large number of children (forty percent of the second years) concentrated on the word "reduced" in the last question dealing with percentages and ignored the sign '%'. They subtracted 5% from £20 to give £15 or divided £20 by 5 and then subtracted the resultant £4 from £20.

The enlargement of an open figure resulted in many children either retaining the gap as it was in the original or they ignored it entirely. This might have been because they saw the 'diagram' simply as a line segment and not as the composition of two line segments aligned in a particular way. The resultant figure obtained when the gap enlargement was ignored did not look the same shape as the original.

Table 1[illegible]. Error Incidence (1977 Sample) (n=743)

Item		Codes 4	5	6	7	8
1b(iii)	2nd					16.6
	3rd					20.2
	4th					23.2
2	2nd					10.5
	3rd					5.1
	4th					4.7
3a(iv)	2nd		11.1			
	3rd		8.9			
	4th		5.3			
3b(i)	2nd		10.1			
	3rd		12.1			
	4th		10.0			
3b(ii)	2nd				8.8	
	3rd				13.6	
	4th				8.9	
3b(iii)	2nd				11.1	
	3rd				9.3	
	4th				8.9	
3b(iv)	2nd				17.2	
	3rd				14.4	
	4th				14.7	
4a V Gap	2nd					13.2
	3rd					17.5
	4th					17.4
4a H Gap	2nd					10.8
	3rd					16.7
	4th					15.8
4b	2nd	9.5		47.6		
	3rd	14.0		37.7		
	4th	10.5		43.2		
5	2nd			49.7		
	3rd			48.2		
	4th			41.6		
6a	2nd					16.6
	3rd					20.2
	4th					21.1
6b	2nd					16.6
	3rd					19.5
	4th					21.6
7a	2nd			37.5		
	3rd			38.5		
	4th			41.6		
7b	2nd			28.0		
	3rd			24.5		
	4th			25.8		
8d	2nd					31.4
	3rd					31.1
	4th					24.2

The errors committed by the 1977 sample (n=743) are shown in table 18. As before only an incidence of more than 10 per cent has been recorded. To a very great extent this second sample of children committed the same errors. Evidence of the addition strategy having been used was found on exactly the same questions as identified from the 1976 data.

The interviews described in this chapter show that children tend to avoid fractions and use additive or 'building-up' methods when dealing with problems in ratio and proportion. The results of the large scale testing show that some errors are committed by many children (from many different schools). Both these aspects of the research have implications for teaching. The next chapter gives details of the results of the testing and the hierarchy formed. The final chapter deals with the teaching implications.

CHAPTER SIX

Analysis of Data and the Establishment of a Hierarchy

The results of the large scale testing are reported in this chapter. The data from the 1976 testing were used to form a hierarchy of understanding in the topic of ratio and proportion. The different statistical methods applied to the data amd the consequent groups of items are discussed in detail in this chapter. The test was also given to another large sample of children in 1977; the results of that testing also appear in this chapter. The 1977 data were analysed in the same way as the 1976 data; the resulting levels of attainment being compared to those obtained from the data of the earlier testing. A comparison is made between the three testings of the children in the longitudinal survey.

Each testing took place in June or July just prior to the long summer vacation. The ratio test was given by the class teacher during the time normally devoted to a mathematics lesson. The marking of the scripts was done according to the marking codes shown in Table 13. Once a hierarchy had been established using the data from the total sample, each child within that sample was assigned to a level in the hierarchy. A comparison of performance according to age was then made. The chapter concludes with a discussion of certain issues that arose out of the results, for example the performance of adults or the effect on performance of rewriting some questions. The Piagetian tasks which have been mentioned earlier and which proved to be inappropriate for the purpose for which they were designed are also discussed.

Facility of Items

In the following tables the word "facility" refers to the percentage of the stated sample of children who had that item correct. The item numbers refer to the test paper which appears at the end of Chapter 3.

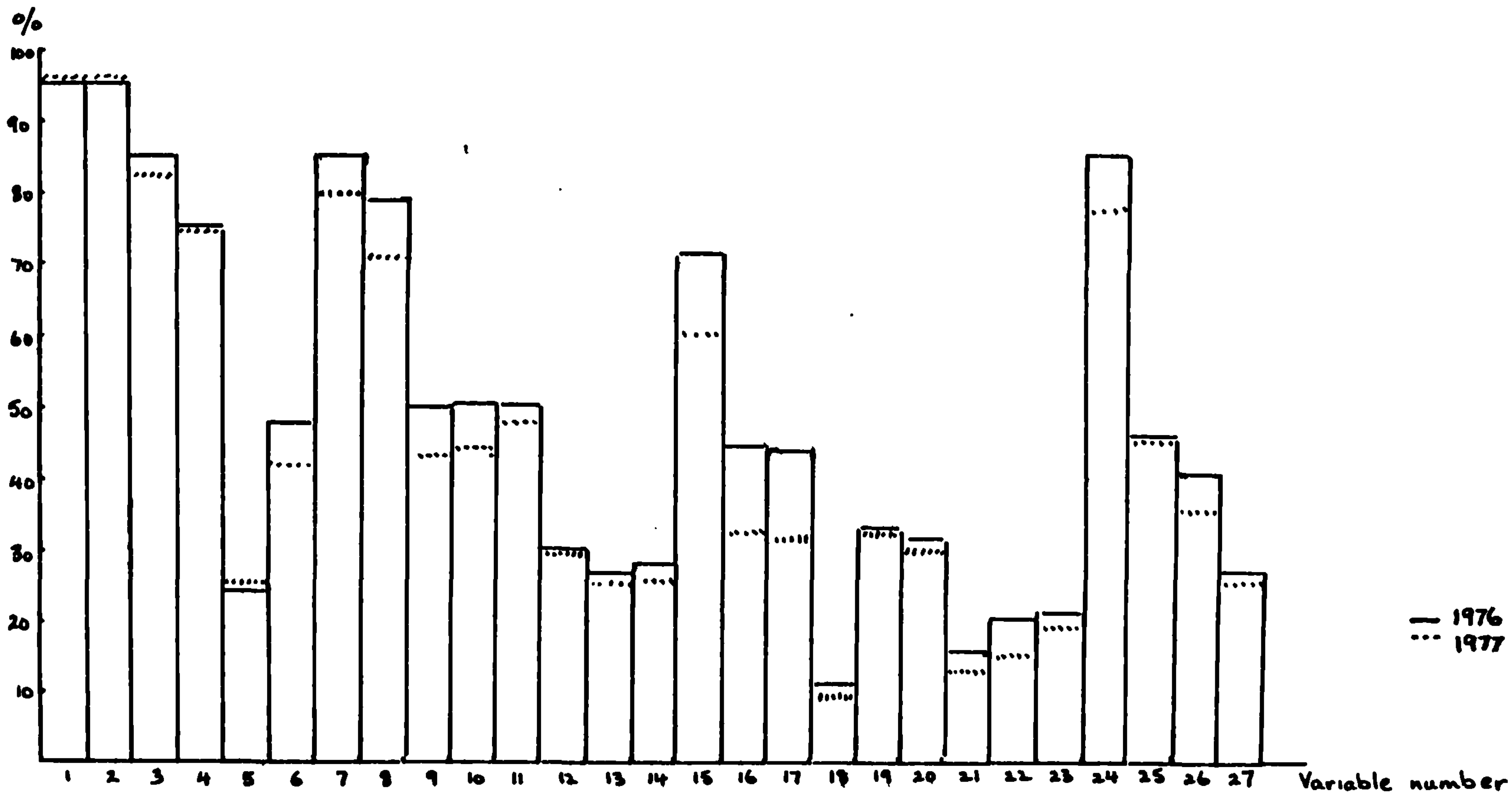
Table 19 shows the facilities of the items in the 1976 and 1977 testings, a graph showing the difference in overall facility (total sample, not year group) appears as figure 7 below. The overall facilities were marginally worse in 1977 although there is a difference of ten percent only on items 4a (i), (ii), (iii). (variables 15, 16, 17). These were the questions dealing with the doubling of a line segment and the gaps in an open rectilinear figure.

TABLE 19. Facilities of Items (1976, 1977 Samples)

Item		2nd yr 1976 n=800	1977 n=296	3rd yr 1976 n=767	1977 n=257	4th yr 1976 n=690	1977 n=190
1a	i)	94.4	93.9	94.3	93.8	95.8	97.9
	ii)	94.6	93.6	95.0	94.2	95.5	98.4
1b	i)	84.5	78.0	84.0	84.0	88.1	85.8
	ii)	75.0	73.0	74.8	74.3	78.6	81.1
	iii)	24.4	20.6	22.7	24.5	31.4	26.8
2		43.8	35.8	46.0	45.8	56.7	43.2
3a	i)	85.4	74.7	83.1	83.7	91.2	84.7
	ii)	75.9	65.5	74.7	75.1	83.3	73.7
	iii)	45.0	35.8	49.5	50.2	56.8	47.9
	iv)	45.6	36.5	50.3	50.6	56.4	50.5
3b	i)	47.0	42.2	47.8	51.4	55.1	54.7
	ii)	22.7	24.3	30.0	34.6	40.6	31.6
	iii)	22.6	19.9	27.4	27.2	32.2	27.9
	iv)	23.7	20.3	28.2	26.8	32.8	28.9
4a	Vert.	70.9	50.3	72.6	63.4	74.1	63.2
	Vert.gap	40.0	27.0	45.1	34.6	46.7	38.4
	Horiz.gap	40.1	26.4	43.3	33.1	44.6	34.2
4b		7.9	5.4	11.0	10.1	19.7	11.1
5		28.1	29.4	29.6	31.1	42.0	35.8
6	i)	27.0	24.7	29.2	33.9	43.2	30.5
	ii)	11.4	10.1	12.3	12.5	18.3	13.2
7	i)	13.7	11.1	19.7	16.7	28.7	19.5
	ii)	15.4	13.9	20.5	17.9	28.8	22.6
8	a)	83.7	68.9	85.7	83.3	85.8	79.5
	b)	37.6	33.8	45.5	48.2	55.7	57.9
	c)	32.0	24.7	39.6	40.1	47.0	45.8
	d)	20.4	16.6	26.5	26.1	35.4	37.9

Note: Code 1 was taken as correct on each item. Table 19 shows percentage correct.

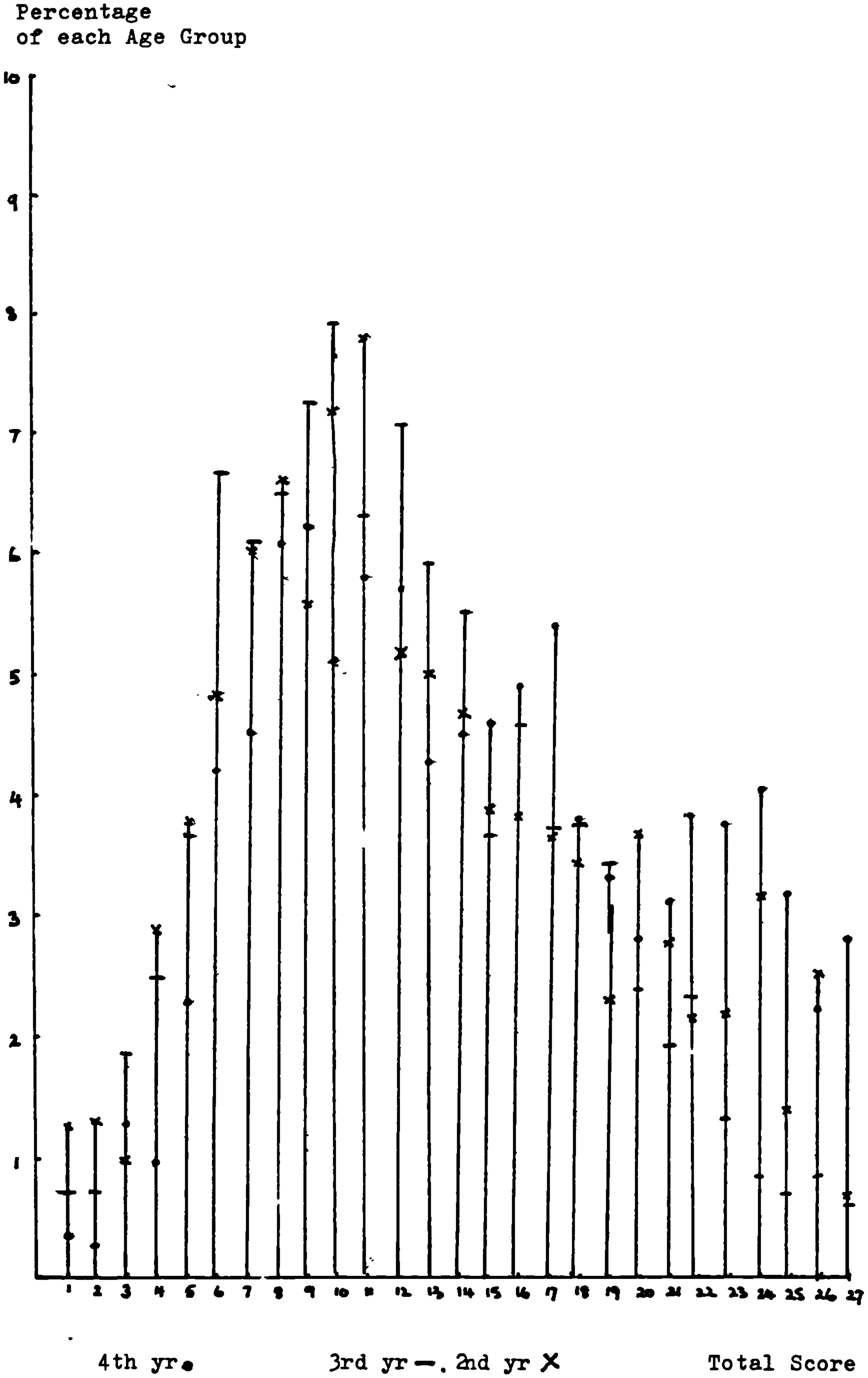
Fig. 7. Comparison of Facilities (1976, 1977 Samples)



The Percentage of Each Age Group

Achieving Total Scores 1-27 (1976 Samples)

Fig. 8



Spread of Facilities

Table 19 details the facility of each item for each year group. Although there were items with high facility and others with low facility, there were no items with a facility value between 72 and 50 percent. The number of items within each ten percent facility band is shown in table 20 below:-

Table 20. The Number of Items Within Facility Bands

(1976 Sample n = 2257)

Facility (percent)	No. of Items
90 - 100	2
80 - 90	2
70 - 80	3
60 - 70	0
50 - 60	0
40 - 50	8
30 - 40	3
20 - 30	5
10 - 20	3

Total Score

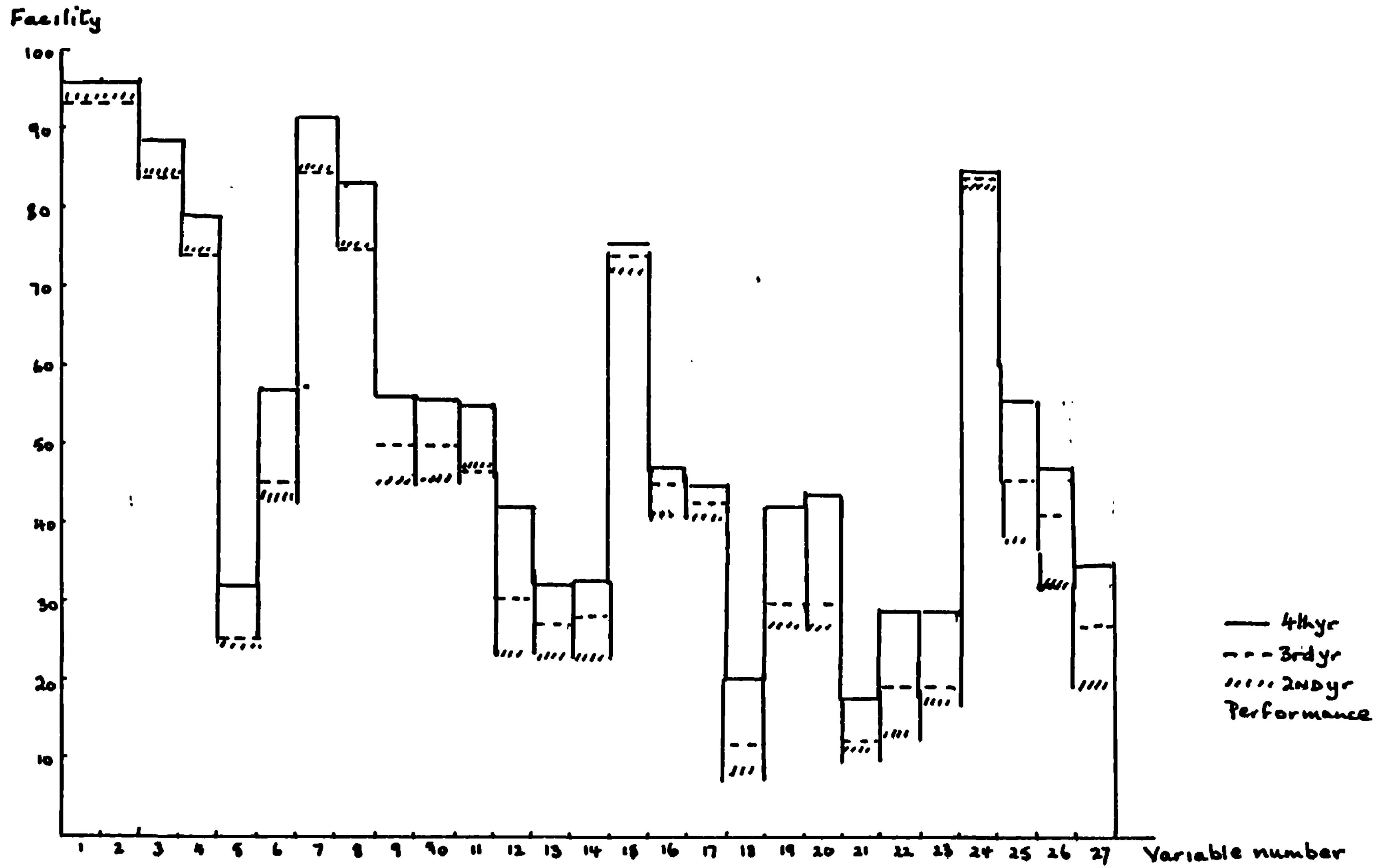
Although each child's performance was to be assessed on the basis of the highest level of the hierarchy which he attained, a rough comparison of year group performance was made using the total score for each child on the 27 items on the test paper. Figure 7 shows the percentage of each age group which attained each of the 27 total scores. It is apparent that many of the fourth formers found the items difficult and that there was little difference in total performance between the years. There were however more children in the fourth year who obtained a high total score than in the other two years see table 21 below.

Table 21. Total Score By Year (76 Sample)

Year	Mean Score	Percentage with more than 22/27 correct
2nd. n=690	12.3 items 45 percent	3.2
3rd. m=767	12.9 items 48 percent	8
4th. n=800	14.65 items 54 percent	13.9

A closer comparison of each age group's performance was made by investigating the percentage success on each item, this is shown in figure 9.

Fig. 9. A Comparison of Performance of Each Item by Year Group (1976 Sample)



From figure 9 we can see that the very easy items were easy for each age level. On some items e.g. variables 25, 26 and 27 (all items dealing with percentage, there appears to be a regular increase in facility in that there is an increase of eight to ten percent for each year tested. On other items e.g. variables 6, 19 and 20 there appears to be an increase in facility only for the fourth year, the scores for the second and third years being very close. It was hoped that these facility patterns would give information to help in the grouping of items. The different facility patterns shown in figure 9 are summarised in table 22.

Table 22 Pattern of Facilities By Year (1976 Sample)

	Items (Variable Nos.)	Facility (percent)
3 years close easy items	1, 2, 15	
2nd and 3rd year close 5-7 percent increase for 4th yr	3, 4, 7, 8	70 - 80
5-7 percent between each yr	9, 10, 25, 26	45 - 55
2nd, 3rd years close 10 percent increase for 4th yr	6, 11	40 - 50
2nd, 3rd years close 10 percent increase for 4th yr	5, 19, 20	25 - 40
5-7 percent between each yr	12, 13, 14, 22, 23, 27	20 - 40
2nd, 3rd years close 7 percent increase for 4th yr	18, 21	15

Summary of Results

The items on the ratio and proportion test paper spanned a facility range from 10 to 90 percent but there was a gap between 50 and 70 percent in which no items appeared. The total scores obtained by the children in each year group did not vary very much, the mean marks being 45, 48, 54 percent. More of the fourth year obtained a high score than children in the other two years. The pattern of success, year by year varied according to the question set. The attempt to form items into groups came later but at this stage three items (variables 24, 16 and 27) seemed to be giving little information on the child's understanding of ratio and proportion. Variable 24 was simply a reiteration of the introduction to the percentage questions and thus needed a simple application of the definition given in the first sentence of the question. Fifteen percent of the children could not do this. Variables 16 and 17 required the child to remember that in the enlargement of an open figure the gaps between the ends of the line segments also had to be enlarged. Some 20 to 30 percent either forgot to enlarge the gap or omitted the gap altogether. This part of the question, although providing information on the child's perception of the figure and its enlargement is strictly not concerned with the understanding of ratio.

Attempts to Form a Hierarchy

The work of other researchers who had used measures of association in order to form groups of items or a hierarchy of items, was described in chapter 4. Many of these methods were used as first attempts at establishing a hierarchy based on the 1976 ratio data. In the following discussion the application of each technique is described prior to the explanation of the method finally adopted.

Listing According to Facility

A simple ranking of items according to facility is insufficient for a hierarchy since although items might be successfully completed by the same percentage of children, those individuals who succeeded on item a) would not necessarily be the same as those who succeeded on item b), even though items a) and b) had the same facility. This certainly could be the case for two items each with a facility value of 30 percent. In the formation of a hierarchy the decision was taken that the following three criteria should apply:-

1) Items should be grouped according to facility
2) Items should be grouped because the same children appeared to do them successfully
3) There should be some link between groups on the easy/hard continuum in that children who were successful on the hardest items should also be successful on the easier ones.

At each stage of the research, features common to various items have been sought, for example the methods used by the children or the mathematical description of the items. The item facilities achieved by each age group were shown in figure 9 and the resulting patterns have been described in table 21. The performance of those children designated 'the adders' in chapter five has also been described for each iten. A further investigation of child performance was made by crosstabulating pairs of items (sometimes triples). The items which gave the most consistent crosstabulation patterns i.e. performance on them did not appear to be random and they were of comparable difficulty were the following:

Variables 12, 13 and 14. Of those who had variable 14 incorrect 79.8 percent also had variables 12 and 13 incorrect. Of those who had variable 14 correct 62.2 percent also had variables 12 and 13 correct.

Variables 6, 9, 10 and 11. Of those who had variables 10 and 11 incorrect 65.8 percent also had variables 6 and 9 incorrect. Of those who had variables 10 and 11 correct 67.9 percent also had variables 6 and 9 correct.

Variables 27, 22, 23 and 13. Of those who had variables 23 and 13 incorrect 82.6 percent also had variables 27 and 22 incorrect. Of those who had variables 23 and 13 correct 62.2 percent also had variables 27 and 22 correct. The cross tabulation of all other sets of items showed no substantial number of children consistently passing or failing.

A factor analysis of all the items was carried out using the varimax rotated factor matrix. The factor loadings on six main factors seemed to be influenced by the facilities of the items. Those items which had a loading of .5 or more on one of the six factors are shown in figure 10. Figure 10 shows possible ways of grouping items using each of the following as a criterion:

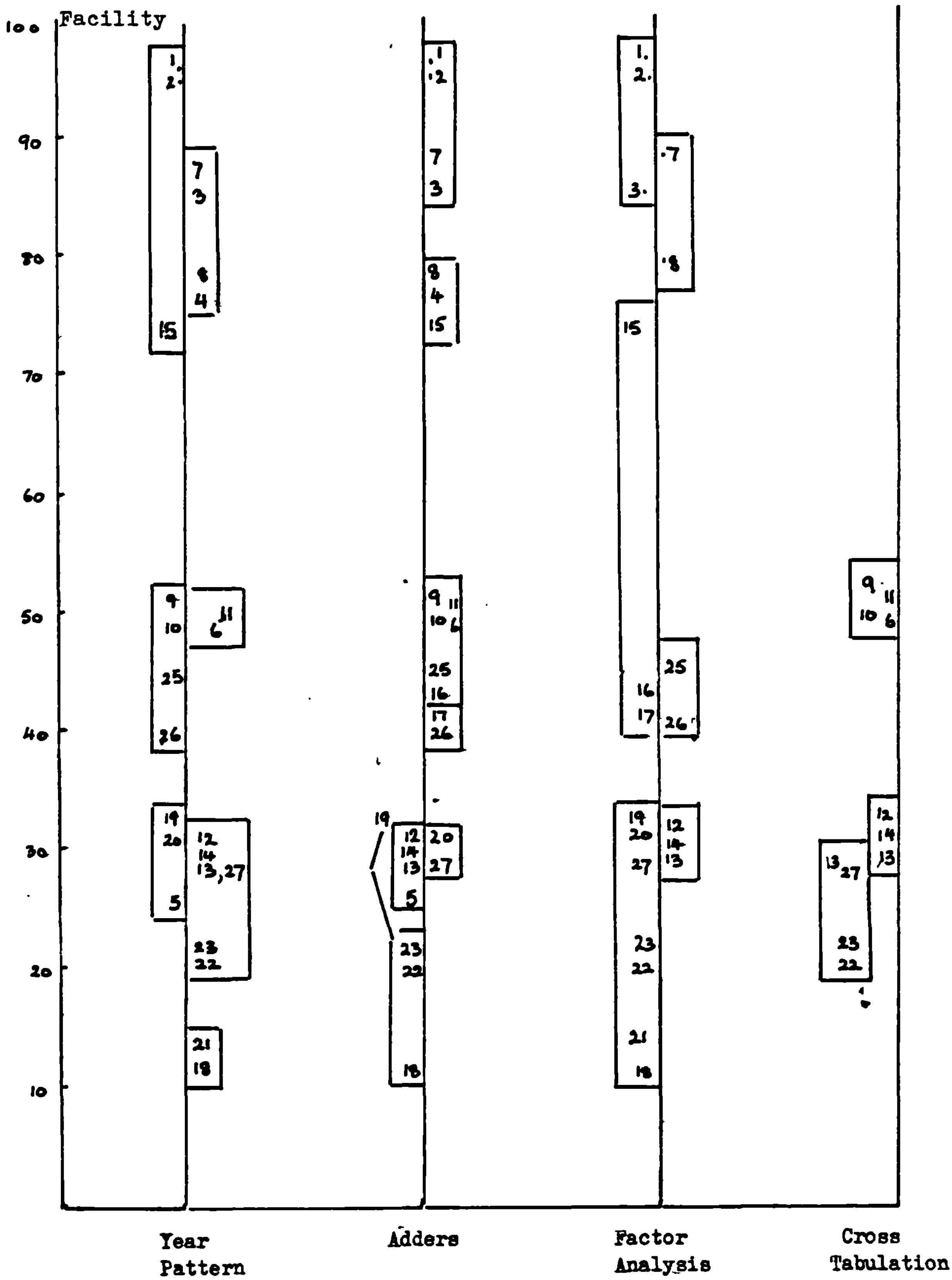
a) Similar age group performance pattern

b) Similar performance by 'the adders'

c) Similar pass/fail patterns from the crosstabulation

d) High loading on the same factor (from the factor analysis)

Fig. 10 Items Groupings . Resulting from Different Methods of Clustering



Note: The numbers are the variable numbers given to the items on the ratio test.

Each method pproduced a different formation of groups and none of the methods described provided information on the link between hard and easy items on the same string of groups. No item had at this stage been omitted from the analysis. A further attempt in which the discrimination of groups of items was assessed was made as follows:

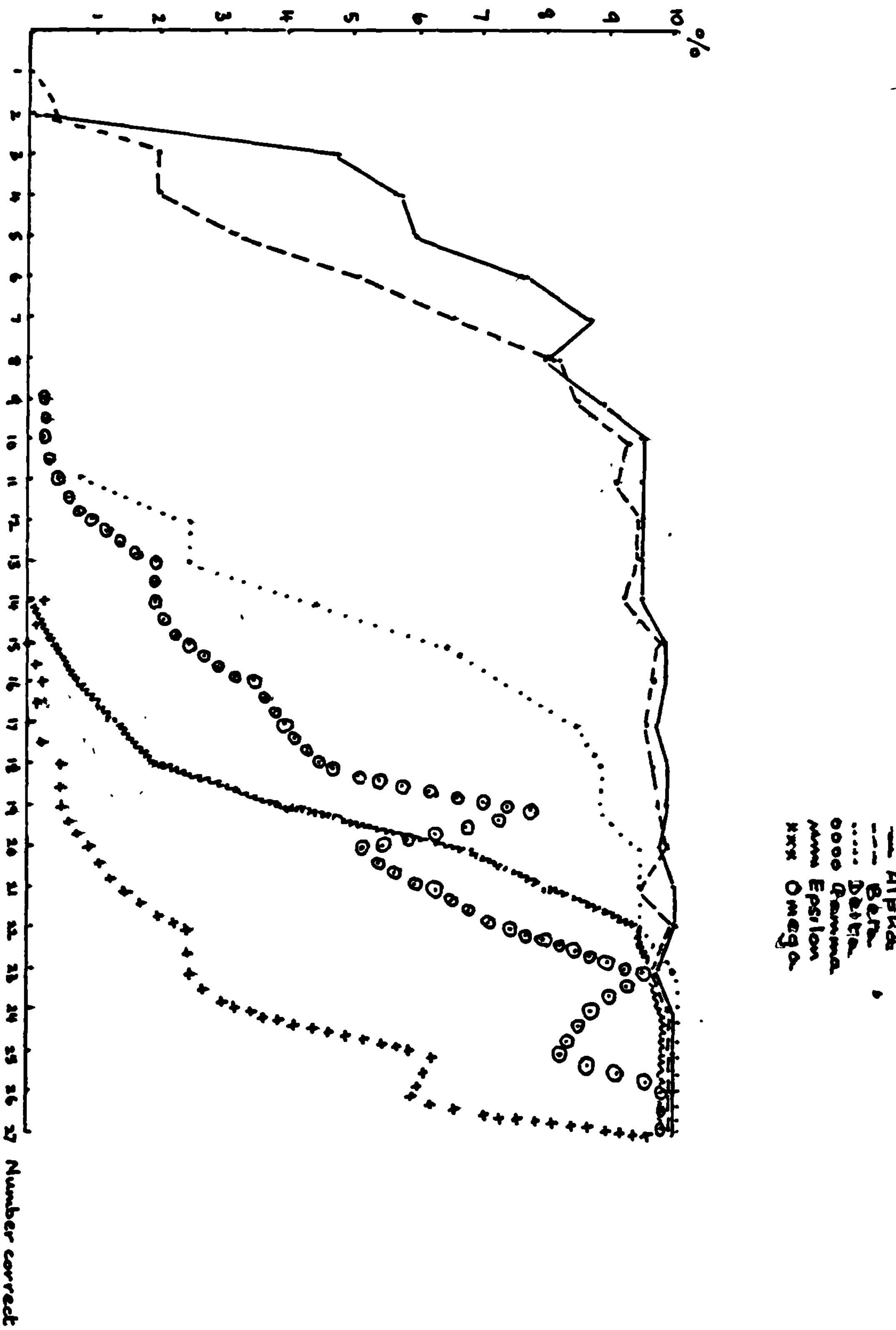
Firstly items with approximately the same facility were put together and then a pass mark of about two thirds correct was assigned, a child would pass the Alpha group for example if he had three of the four items correct. In order to find the measure of discrimination of each group, the number of children who passed each was matched against their total score. Figure 11 shows the result, for example the percentage of children with a total score of ten who passed group Alpha is 95.

Table 23. Groups According to Facility, Alpha - Omega

Group	Variables	Pass Mark Assigned
Alpha	1, 2, 3, 4,	3/4
Beta	7, 8, 15	2/3
Gamma	16, 17, 19, 20	3/4
Delta	6, 9, 10, 11, 25, 26	4/6
Epsilon	27, 22, 23, 12, 13, 14, 21	5/7
Omega	18,5	1/2 and 2/2

Figure 11 shows the degree of discrimination of each group when matched against total score. Group Gamma did not discriminate well, one would expect a profile similar to that of the Delta group for good discrimination. A pass mark of 1/2 for the Omega group was obviously inadequate, the 2/2 pass mark gave a better prpfile.

Fig 11. Discrimination of Groups Alpha - Omega



The groups Alphs - Omega were subjected to a Guttman scalogram analysis, the pass marks being those stated in table 23 (2/2 being the pass mark for group Omega). The results of the scalogram analysis were as follows:

Errors 1414 n = 2257

Coefficient of Reproducibility 0.8956

Coefficient of Scalability 0.5641

The number of errors was very large and it was obvious that the six groups did not satisfy the criterion that success on the harder groups of items entailed success on the easier groups. The method of grouping just described did not use information regarding the children who did the same questions. Methods of forming groups which take into account success by the same children are described below.

Grouping Items Using Measures of Association

The use of homogeneity coefficients has been discussed with reference to the work of other researchers, (see chapter four). In this chapter the coefficients are applied to the ratio data. The facility of each item was available and the problem was to form groups of items. Homogeneity coefficients which used the fourfold table classifying n individuals according to their performance on two items I_1 and I_2, seemed suitable for grouping items. The indices considered were ϕ, H_{ij} and Q, all of which are defined below.

Items I_1 and I_2

	Fail I_1	Pass I_1		
Pass I_2	a	b		
Fail I_2	c	d		a, b, c, d no. of children
			n	n total number of children

in this notation the indices are:

$$\phi = \frac{bc - ad}{((a+b)\ (c+d)\ (a+c)\ b+d))^{\frac{1}{2}}} \quad \text{(Guilford 1965)}$$

$$H_{ij} = \frac{bc - ad}{(b+d)(c+d)} \quad \text{(Loevinger 1947)}$$

$$Q = \frac{bc - ad}{bc + ad} \quad \text{(Yule 1912)}$$

(In the calculation of H_{ij}, I_1 must be the harder item i.e. d<a)

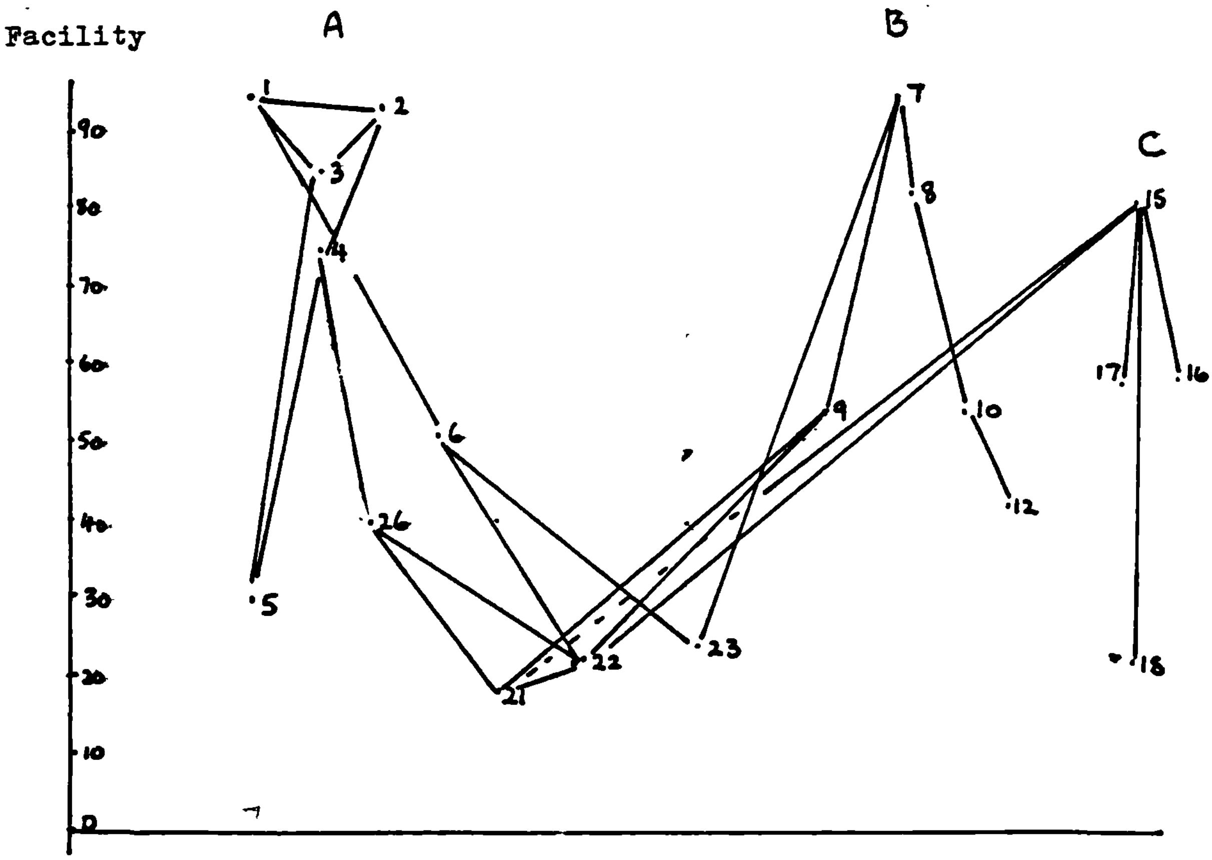
Grouping Items Using Loevinger H_{ij}

Perfect homogeneity i.e. a coefficient of one is seldom found when real data are used, an acceptable value of the coefficient has therefore to be decided. The coefficient was used in the following way:

1) Item - item Loevinger coefficients were found for all pairs of items.
2) The facility of the easiest item was plotted on graph paper and all items which had an H_{ij} of more than .64 with this item were also plotted.
3) Other easy items and those additional items which were connected to them by a high value of H_{ij} were also plotted.
4) All other connections resulting from a high value of H_{ij} were plotted.
5) Since groups of items at approximately the same level of facility were being sought, items which were connected only to others of very different facility were rejected.

The above procedure resulted in the rejection of a number of items. The items plotted did not cluster in groups within facility levels except at the harder than fifty percent level, see figure 12.

Fig 12. The Loevinger Coefficient H_{ij} Applied to Ratio Data (1976)

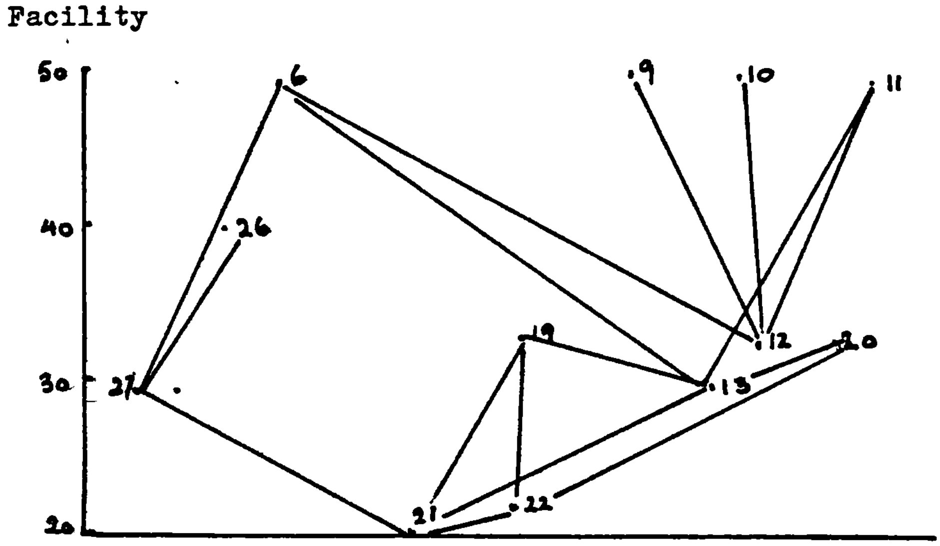


H_{ij} = .65. The numbers shown are variable numbers.

The values of the coefficient obtained when the two items being compared were both easy, were very low. However if the criterion value of H_{ij} was made less stringent certain items within the 20-50 percent range could be connected. A connection simply between items of the same facility was insufficient if the aim was to form a hierarchy, therefore when the coefficient was lowered to .55 it was decided that the items included on the diagram had to be connected to both the very easy items and some of the hardest.

The connections between items using these new criteria are shown in figure 13 below.

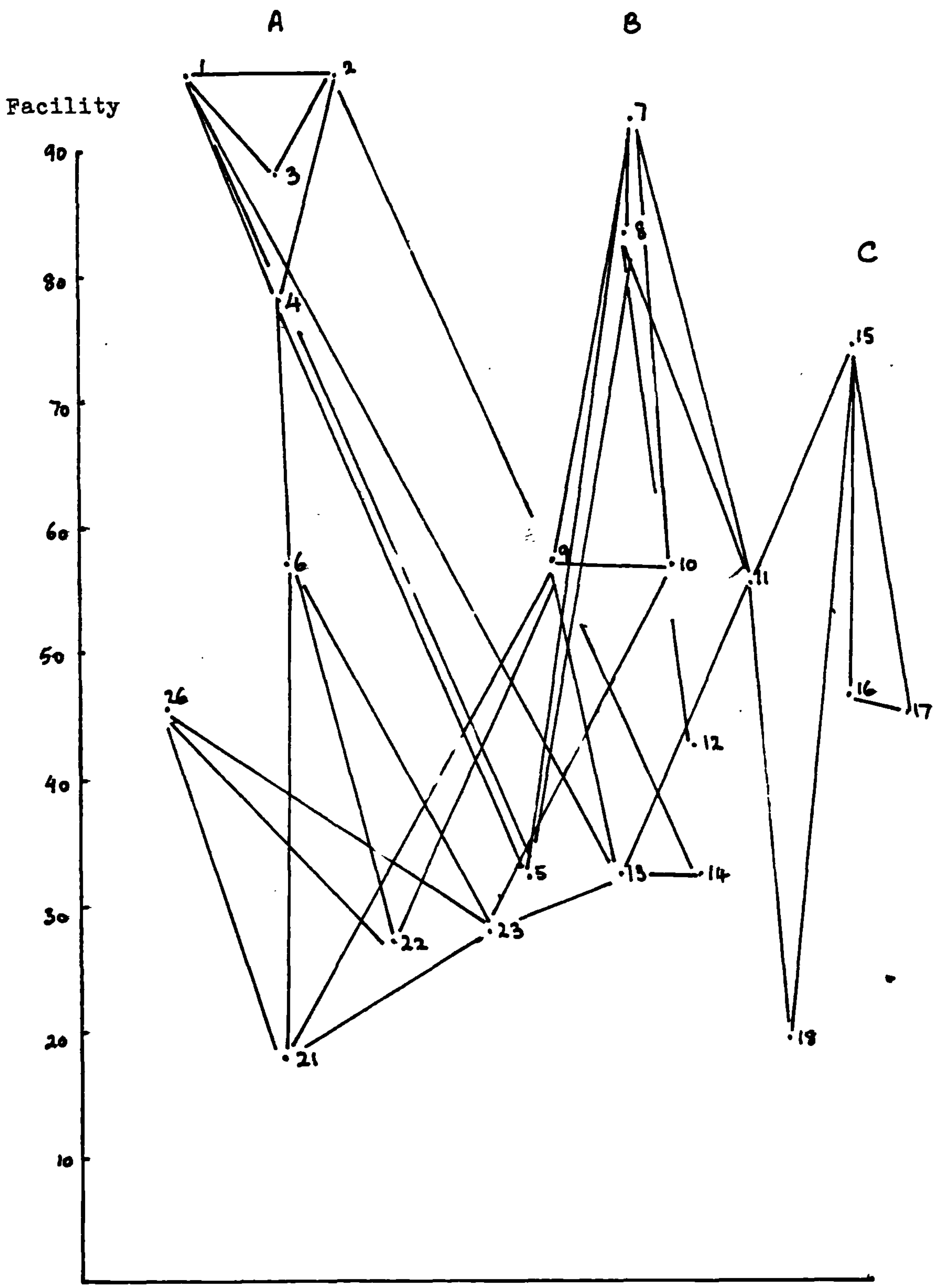
Fig. 13. Using Loevinger Coefficient (H_{ij} = .55) on Ratio Data



Note. Variable numbers are shown.

Nassefat (1973) argued that a greater degree of homogeneity between tasks was apparent when the children doing the tasks demonstrated a degree of stability in the cognitive level demanded by those same tasks. Extending this idea it could be argued that the data from the oldest children tested on the ratio paper should produce the greatest values of the homogeneity coefficient item/item. Figure 14 shows the connections between items when the results of only the fourth year children were used; $H_{ij} \geqslant .64$.

Fig. 14. Using Loevånger Coefficient ($H_{ij} \geqslant .65$) 4th Year Data



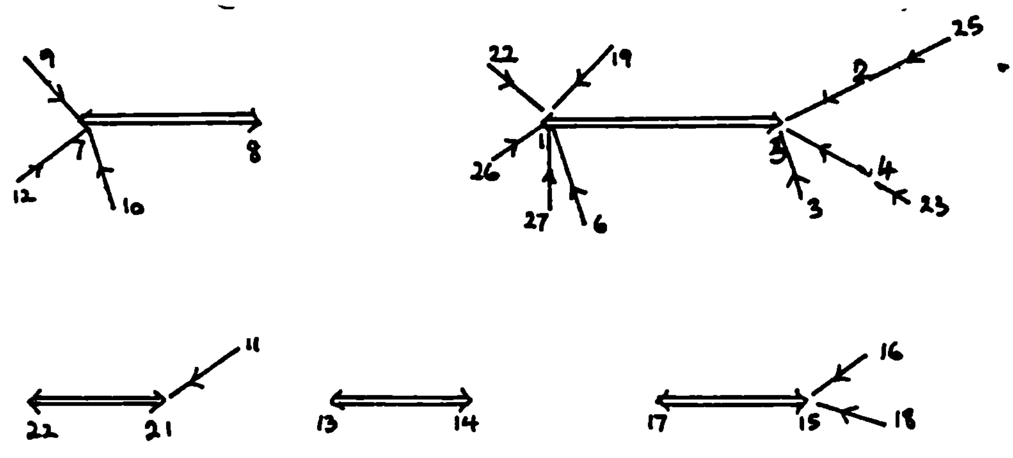
Note. Variable numbers are shown.

All the above attempts to establish a hierarchy used the Loevinger coefficient H_{ij} with various restrictions. Other researchers (McQuitty, Bart and Krus) had used homogeneity coefficients in a different way; the methods used by these researchers were applied to the ratio data: Linkage Analysis (McQuitty 1957)

McQuitty used a comparison person to person in order to describe types of people. Applying his method to the ratio data one firstly listed all the item/item coefficients (Loevinger H_{ij}) and underlined the highest value of H_{ij} for each item. The two items which possessed the highest value of H_{ij} formed the first 'type'. In the coefficients from the ratio data these were variables 7 and 8. All items which had their highest coefficients with variables 7 or 8 were then appended. The process was continued to include those items which had their highest coefficients with an appendage. When type one was exhausted the pair of items with the highest coefficient as yet unused formed the second type and the process was continued. The ratio data coefficients showed five 'types', these are shown in figure 15.

A drawback of the McQuitty method is that two items may be of the same type although they have a low value of the homogeneity coefficient as there is no examination of the relative values of the coefficients.

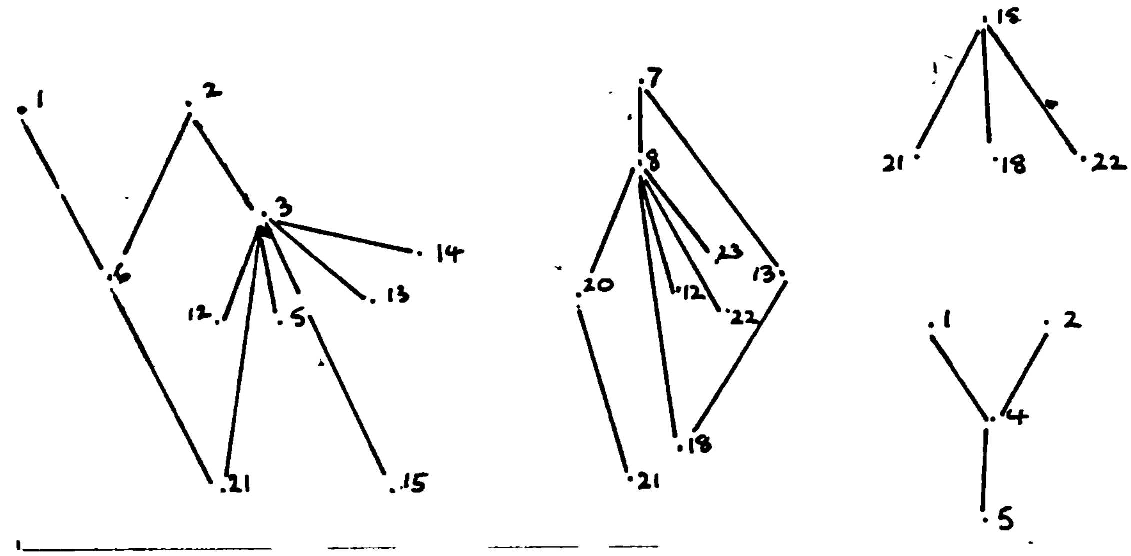
Fig. 15. Grouping Items by McQuitty Linkage Analysis.



The Analysis of Bart and Krus (1973)

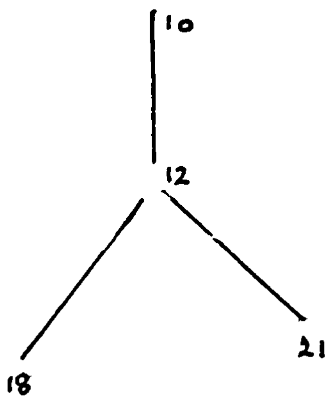
Bart and Krus proposed another form of analysis, again involving a vertical chain of items hard/easy rather than facility grouping. They investigated two items i and j, if (0,1) meant (incorrect, correct) their interest was in the cell of the four cell matrix which displayed item i-0, item j-1. Item i was called a prerequisite for item j if a zero appeared in the cell, i.e. the cell was empty, which showed fail item i, pass item j. When the ratio data were analysed, no cell in the item/item matrices contained a zero entry. Entries close to zero had to be accepted, a cell entry of 2 percent of the sample was called a 2 percent tolerance level. At the two percent level variables 1 and 2 were prerequisites for every other item except variables 7, 8 and 15. At the same tolerance level variable 3 was a prerequisite for nine items. Figure 16 shows the trees which were apparent when the two percent tolerance level was applied. The only tree with more than three stages is that which connects variables 7, 8, 20 and 21.

Fig. 16. Bart and Krus Method. Two Percent Tolerance.



When a five percent tolerance level was used, the distinction between the easiest items was lost and variables 1 and 2 became prerequisites for variables 7, 8 and 15. Very nearly all the items were prerequisite for variables 18 and 21 and many were prerequisite for variables 22 and 23. Linkages between items of medium facility were the most interesting but only one was apparent, see figure 17.

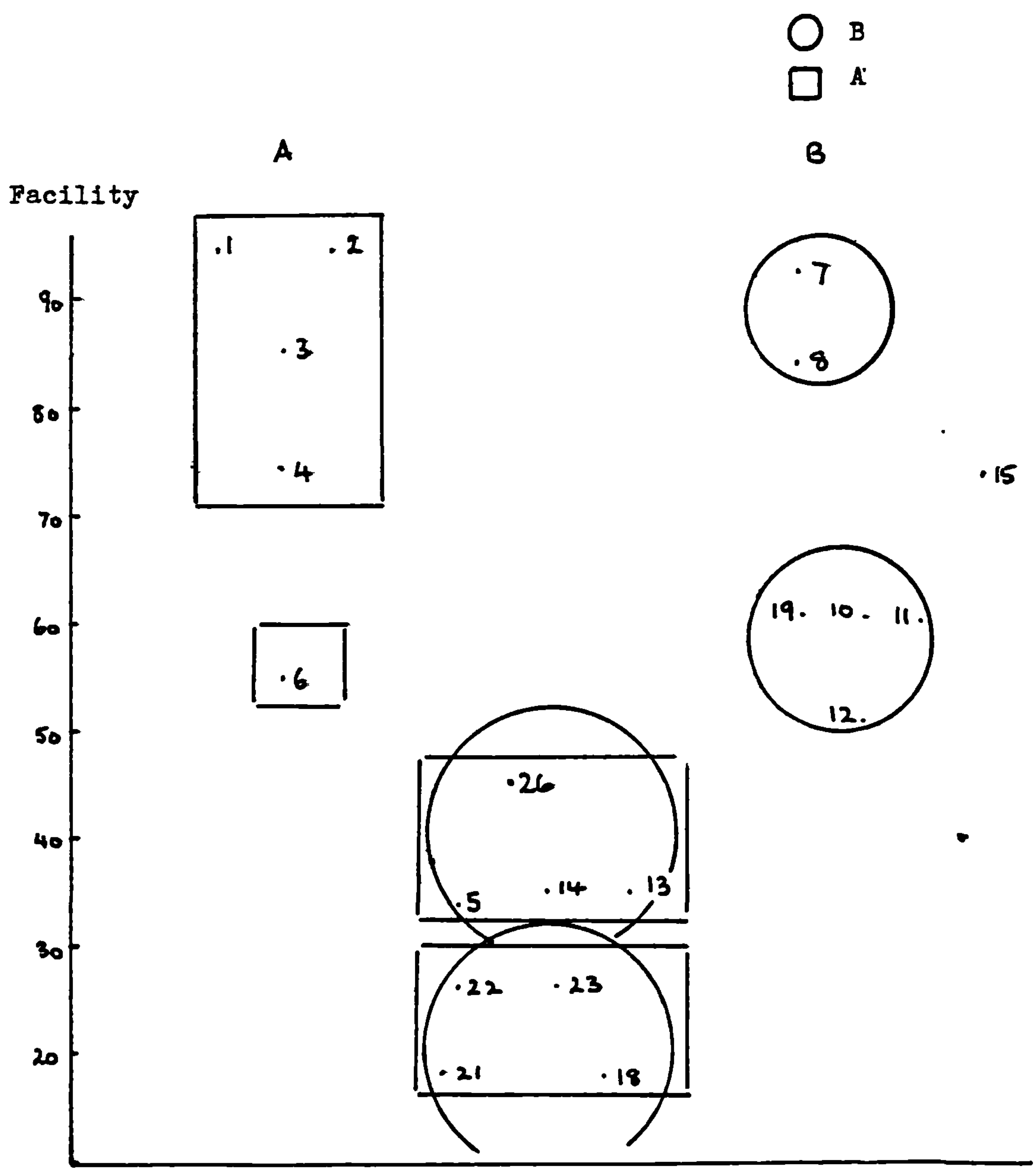
Fig. 17. Bart and Krus Method; 5 Percent Tolerance.



All the above methods when applied to the ratio data produced links between hard and easy items but few connections between items of the same facility. The easiest items had no strong links and there appeared to be three strands A, B and C as shown in figure 12. McCready and Merwin (1973) had addressed themselves specifically to clustering items within a facility band. They investigated 'item forms' with regard to diagnostic domain referenced tests. An item form was considered inadequate if a) the items within the form were not homogeneous, b) the items were not of equivalent difficulty or c) both of these. To study the nature of the relationship among items within an item form they used Loevinger's index of homogeneity H_t (Loevinger 1947), this index is defined on page 110. They took $H_t \geqslant .5$ as an indication of relative homogeneity. When the ratio data were used, it was found that H_t could equal .5 when two items within a group of three had a high value of H_{ij} (item/item), the third item being connected at a low level with each of them.

The method outlined by McCready and Merwin was applied to the data obtained from the testing of the fourth years. It was applied only to items which occurred in strands A and B since strand C contained only three items, two of which were known to have caused difficulty (the gaps in the open figure question). The values of H_t for the groups appear in tables 24 and 25.

Fig. 18. Groups Suggested from 4th Year Data Using Method of McCready and Merwin



Note. Numbers shown are variable numbers.

Table 24. Using Loevinger H_t Values for Grouping (4th Year Data)

Group	Variables	Value of H_t
1 A	1, 2, 3, 4	.668
1 B	7, 8	.98
2 B	9, 10, 11, 12	.537
3 A, B	5, 26, 14, 13	.569
4 A, B	21, 22, 23	.628
Alternative group 4	21, 22, 23, 18	.566

Note. All the groups possess an H_t level within the stated criteria.

Table 25. Using Loevinger H_t Values for Grouping (Total Sample)

Group	Variables	Value of H_t
1 A	1, 2, 3, 4	.625
1 B	7, 8	.96
2 B	9, 10, 11, 12	.506
3 A, B	5, 26, 14, 13	.505
4 A, B	22, 23, 21, 18	.501

The sets of items were chosen according to facility and H_t found. The values of H_t when the total sample was used were lower than those for the 4th year sample although always over .5. In the above tables the groups are labelled by the strand in which they occur, if one assumed that the two groups of easiest items were distinct. An attempt was made to include item 6 in group 2B but this resulted in a value of .454 for H_t.

When the items on the ratio test paper had been grouped by using H_t as just described, the children who had formed the sample were assigned to a level of understanding on the basis of the hardest group of items they passed. This entailed the provision of a pass mark for each group of items and the institution of a stage zero which preceded stage one on either strand and meant that the child did not obtain a group one pass. A child was assigned to level 2B if he achieved a pass mark on levels 1 and 2 in the B strand, any partial success on strand A was ignored. If he was assigned level 3 he could have been successful on levels 1 and 2 in strand A or levels 1 and 2 in strand B. In order to test scalability of the groups within each strand i.e. to test that success on a harder group entailed success on all easier groups, a Guttman scalogram analysis was carried out on each strand. Two pass marks for each group of items were tried. The results appear in table 26 below.

Table 26. Guttman Scalogram Analysis Used on Two Strands (1976 Data)

Sample	Strand	Pass Mark Gp 1	Gp 2	Gp 3	Gp 4	Coefft. of Reproducibility	Coefft. of Scalability
Total	A	3/4	1/1	3/4	4/4	.9759	.8848
n=2257	A	3/4	1/1	4/4	4/4	.9814	.9001
	B	2/2	3/4	3/4	4/4	.9688	.8494
	B	1/2	3/4	4/4	4/4	.9854	.9088
2nd year	A	3/4	1/1	3/4	4/4	.9781	.8803
n=800	B	2/2	3/4	3/4	4/4	.9706	.8365
4th year	A	3/4	1/1	3/4	4/4	.9701	.8667
n=690	B	2/2	3/4	3/4	4/4	.9360	.8431

By taking each strand separately for the scalogram analysis the error count on strand B3 (i.e. success on level 3 without success on all previous levels in that strand) would include those children who reached level 3 by the A route. The true error children were found to be 33 in number. Table 27 shows the performance of the total sample, the child was assigned to the highest level at which he obtained the pass mark, whether it was on strand A or on strand B. The less stringent pass marks shown in table 26 were used. Those children who obtained pass marks on both strands appear under the heading 'Both'.

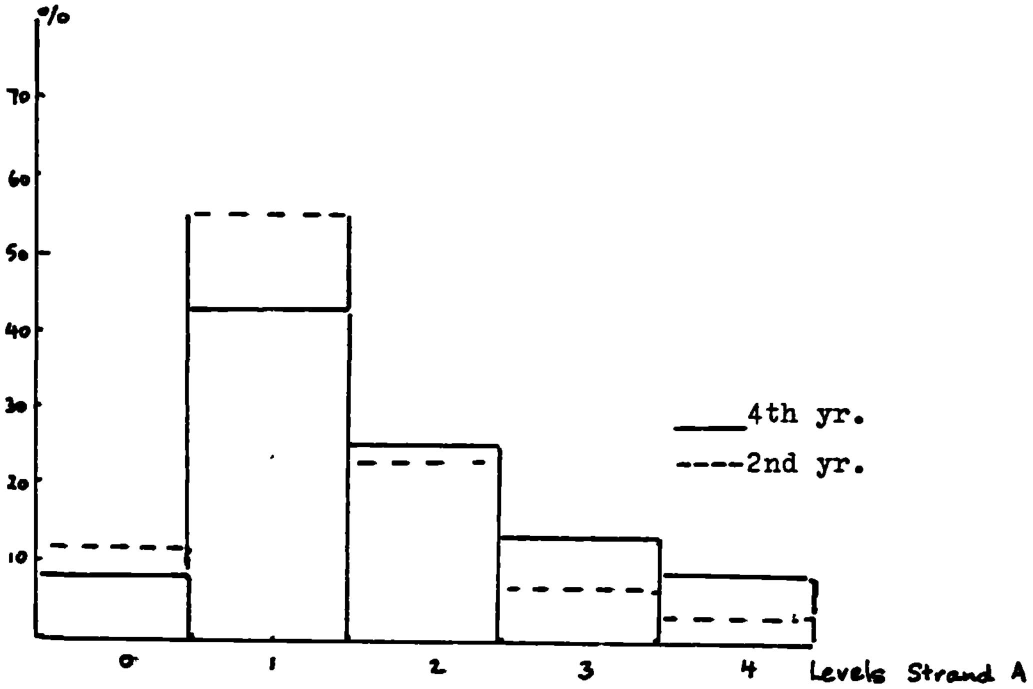
Table 27. Performance on Each Strand (Total Sample n=2257)

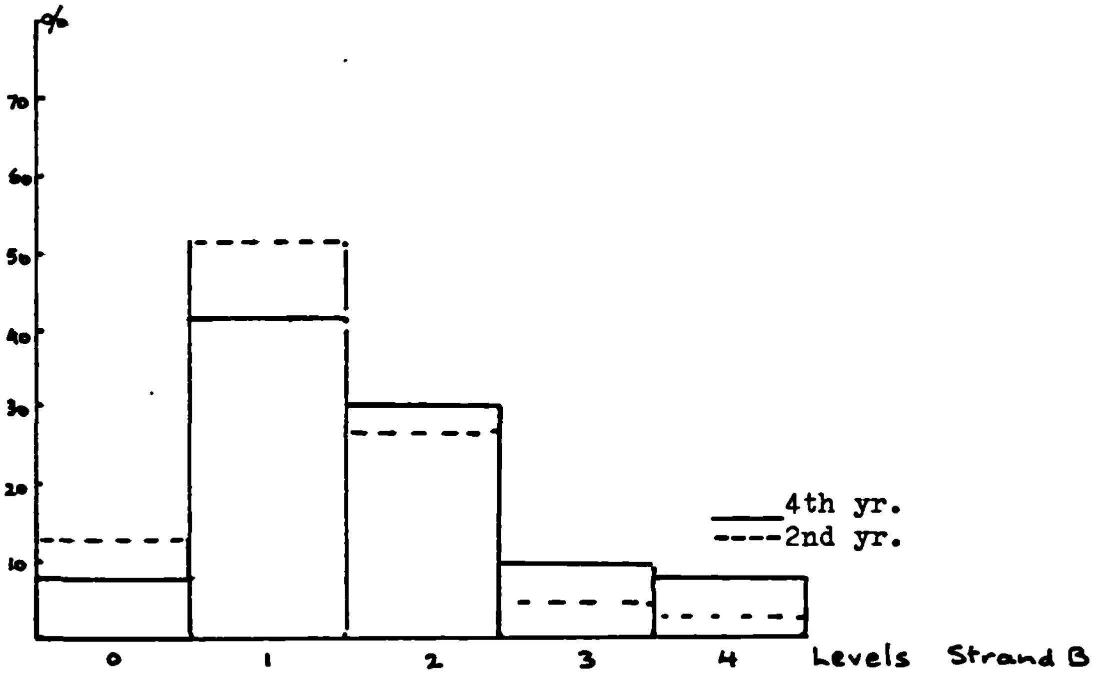
Strand A	Both	Strand B	
	92		Level 0
282	488	113	Level 1
403	245	158	Level 2
71	246	38	Level 3
6	73	3	Level 4

Error children - 33; 13 - 4th yr; 12 - 3rd yr; 8 - 2nd yr.

Although the great majority of children who obtained the pass mark on the hardest group of items had managed to obtain the requisite passmarks on all previous groups on both strands, it was possible to score on the hardest items when missing some of the easier items on one of the strands. A comparison of the performance of the second and fourth years on each strand is shown in figure 19.

Fig. 19. Performance of 2nd and 4th Years on Strands A and B





The Problems Arising From These First Efforts to Group Items

The last analysis described, grouped the items within facility bands but provided two alternative routes for the child to reach success on the hardest items. A child could be at level 1 on strand A or at level 1 on strand B but there was no information which would allow one to equate the two levels. In addition level 2 on strand A consisted of just variable 6, failure on this one item meant a relegation to the previous level. Some of the groups in the strand analysis were composed of variables which were parts of the same question; the homogeneity of these variables might have arisen solely from this fact.

One of the purposes of the C.S.M.S. research was to be the foundation of a network of levels of understanding in many different topics. The problem of whether level 1 in strand A was equivalent to level 1 in strand B would be exacerbated if other hierarchies also had strands and it would seem that a cross match between topics would be very difficult. Since the separation into strands was based on the criterion value of the homogeneity coefficient taken, it was feasible that choosing a lower value of that coefficient would do away with the necessity for two strands. In addition this would enable more items to be formed into a group. The decision was therefore taken that the items from each test paper should be grouped to form a single hierarchy and that strands should be ignored. Thus the delineation of one group at any one facility level was the first priority.

All the statistical methods so far described involved the use of one of the two Loevinger coefficients of homogeneity, H_{ij} or H_t. Other coefficients were available and one of these, $\emptyset$ proved to be of greater use when the items under consideration were of approximately the same facility. The values of H_{ij} obtained from the ratio data were always highest when the two items being considered were very different in facility, this resulted in the first links being made between items which were not in the same facility band.

Considering the homogeneity coefficients H_{ij}, Q and $\emptyset$ mentioned earlier let us suppose the two items are heterogeneous, that is passing one item does not affect the probability of passing the other. In this case all three indices have an expected value of zero. Negative values of all these coefficients are rare in the normal testing situation, since it would be rather unusual to include two items, which were inversely related, in the same test. As used here, the principal function of the indices would be to differentiate degrees of positive relationship.

Q and H_{ij} both have maximum value of +1, regardless of the perceived difficulties of the items and the maximum is attained when passing the harder item implies passing the easier. Phi, on the other hand, can attain its absolute maximum value of +1 only when the perceived difficulties of the items are the same, and in general has a maximum value which decreases as the difference in difficulty between the two items increases.

Indeed one formulation of H_{ij} is as $\frac{\phi}{\phi_{max}}$, where ϕ_{max} is the maximum attainable value of ϕ given the perceived difficulties of the items, and this fact supports the use of H_{ij} when a measure of the homogeneity of two items, very different in facility, is required. However, the constraint on the maximum value of ϕ would tend to support its use, when the aim is to detect groups of items, which are homogeneous and have similar level of difficulty.

<u>Table 28. Illustration of the Use of Three Homogeneity Coefficients</u>

	Example <u>1</u>				Example <u>2</u>		
	Fail	Pass			Fail	Pass	
Pass	99	1		Pass	22	39	
Fail	1	0	101	Fail	38	2	101
$\phi = 0.01$				$\phi = 0.59$			
$H_{ij} = 1.$				$H_{ij} = 0.88$			
$Q = 1.$				$Q = 0.94$			

In the first example a hard item has been compared with an easy one. The probability of the outcome shown for two heterogeneous items, is 0.99 and thus there is almost <u>no support for</u> homogeneity. H_{ij} and Q, however, have high values because there is <u>no evidence against</u> homogeneity, in that no-one has passed the harder item without having passed the easier. In the second example there is more support for homogeneity, the probability of an outcome as extreme as this being less than 0.001 for two heterogeneous items. ϕ has correspondingly increased, while H_{ij} and Q have fallen.

In fact, the absence of a high H_{ij} value between a very easy and a very difficult item may be more noteworthy since it implies that children who have passed the harder item do not have an improved chance of success on the easier. Clearly, any analytical procedure based on H_{ij} or Q would emphasise those relationships of least interest, namely those between very difficult and very easy items. In an agglomerative cluster analysis method, for example, if H_{ij} or Q were used then, very early on, hard and easy items would combine. This would tend to obscure any more informative clusters. Use of phi, on the other, would leave hard and easy items ungrouped at first.

The Formation of a Hierarchy

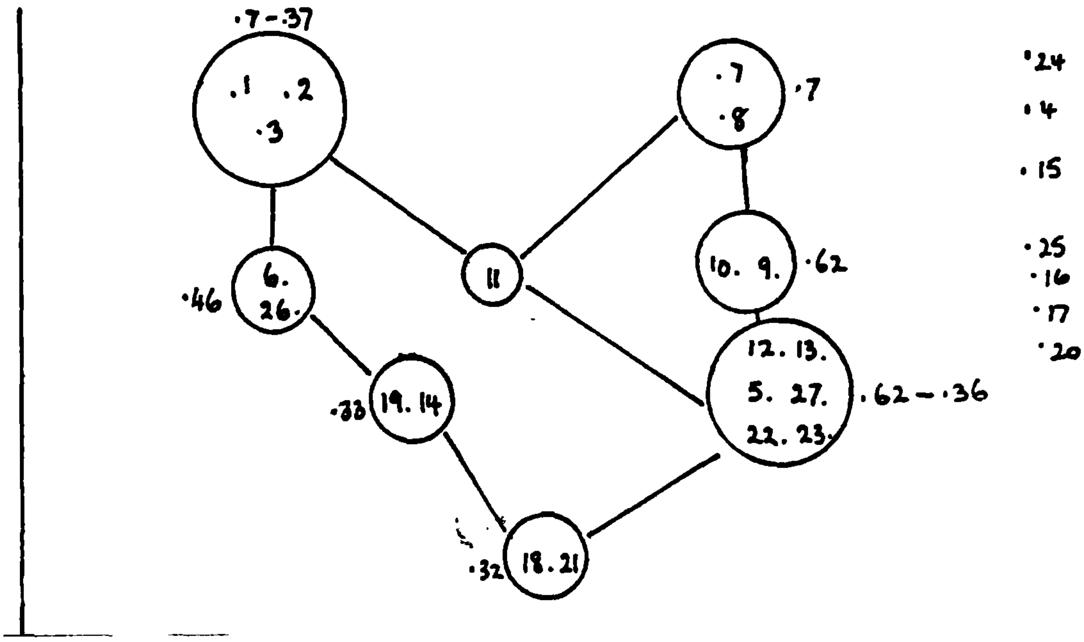
The preliminary attempts to form a hierarchy using the ratio data have already been described, the two final methods were 'the plotting method' and 'complete linkage'. Both methods made use of the coefficient $\emptyset$, the plotting method also took into account values of the Loevinger coefficient H_{ij} for items of very different facility since the absence of a high value of H_{ij} in these circumstances would infer non-scalability. It was decided that since groups of items (the set of groups forming a chain) was needed then the following criteria should be applied. Each group should:

1) Contain items of similar facility
2) Show some association item/item based on child performance
3) Be scalable group/group

The Plotting Method

All the items were plotted on graph paper against an axis labelled with facility, then lines were drawn to connect all items which possessed a $\emptyset$ of above .6. Next lines were drawn to represent $\emptyset$ = .5, .4 and .3. There were some items which connected at a low level (.3) with many items, others which connected at a high level with only one other item. There were also of course items which did not connect at $\emptyset \geqslant .3$ with anything, these were for the moment omitted. An additional criterion for inclusion was that the item should possess a value of $H_{ij} \geqslant .6$ with at least one of the items of very different facility. The items that remained after these two conditions were imposed are shown in figure 20 below. Any item which connected with only one other item was regarded as peripheral to the general structure or skeleton of the test. These peripheral items are labelled with their variable numbers and appear on the right hand side of figure 20.

Fig. 20. Analysis of Data Using Plotting Method



Note. The numbers refer to the variable numbers that appear on the final form of the test paper. An indication of the Ø values between items in the circles is shown and the strongest hard/easy links (Ø or H_{ij}) are shown by lines.

Facility Ordering Complete Linkage

A criterion value of phi was first chosen. Starting with the easiest item, the next most easy item with a phi value greater than the criterion was joined to it. The third item to join the group was the next most easy item with phi values greater than the criterion with both items already included. Subsequent items were joined similarly, provided the phi values with the items already included were all greater than the criterion. This continued until there were no further items achieving the criterion and the process was then repeated, starting with the second most easy item. Thus a chain of items was attached to each item. Some of the chains were subsets of others and could be ignored, overlaps had to be examined further.

In the ratio test the chains formed by the procedure outlined above were plotted against observed facility level and are shown in figures 21, 22.

'ig. 21. Analysis of Data Using Linkage $\phi \geq .40$

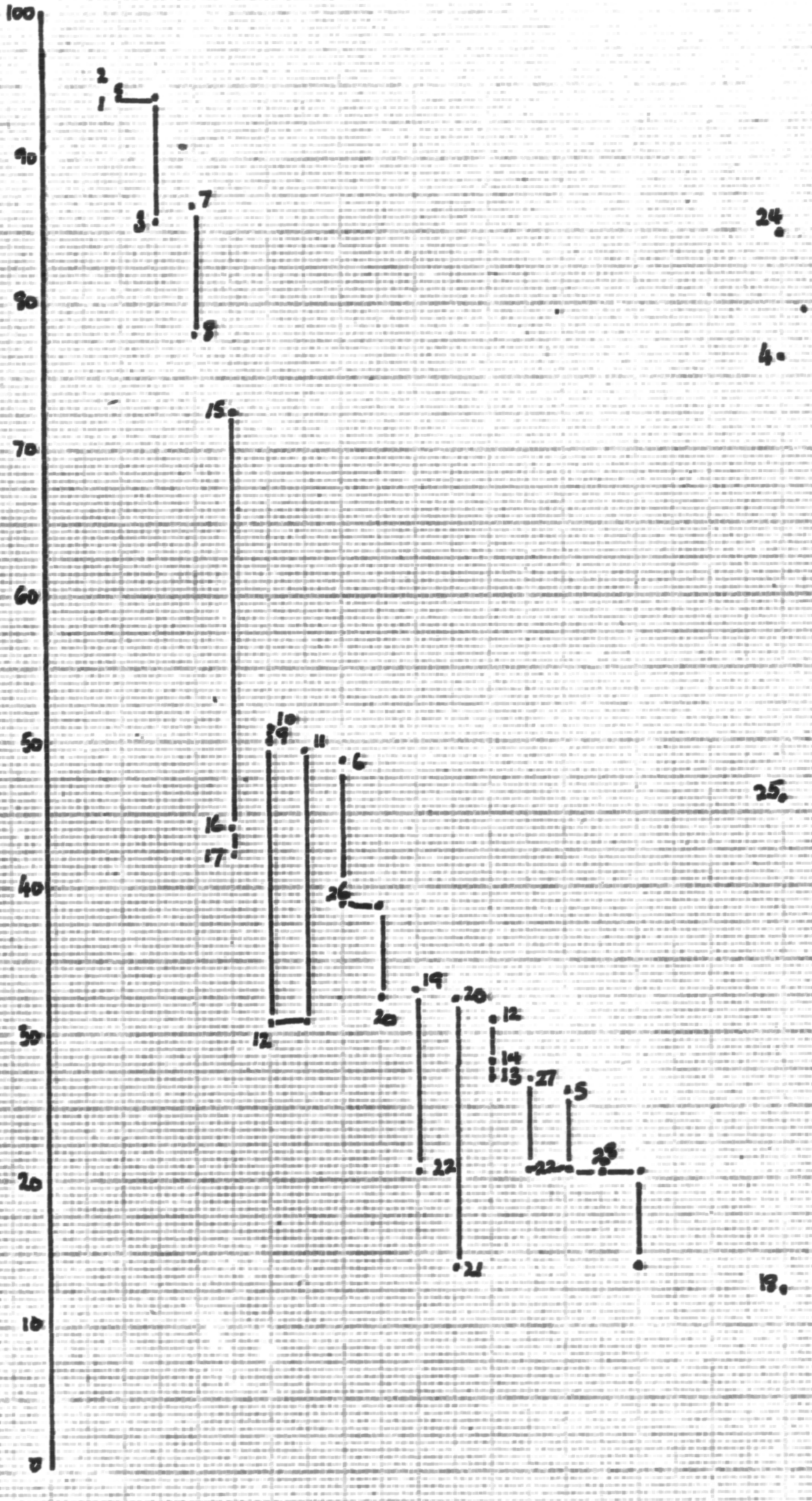
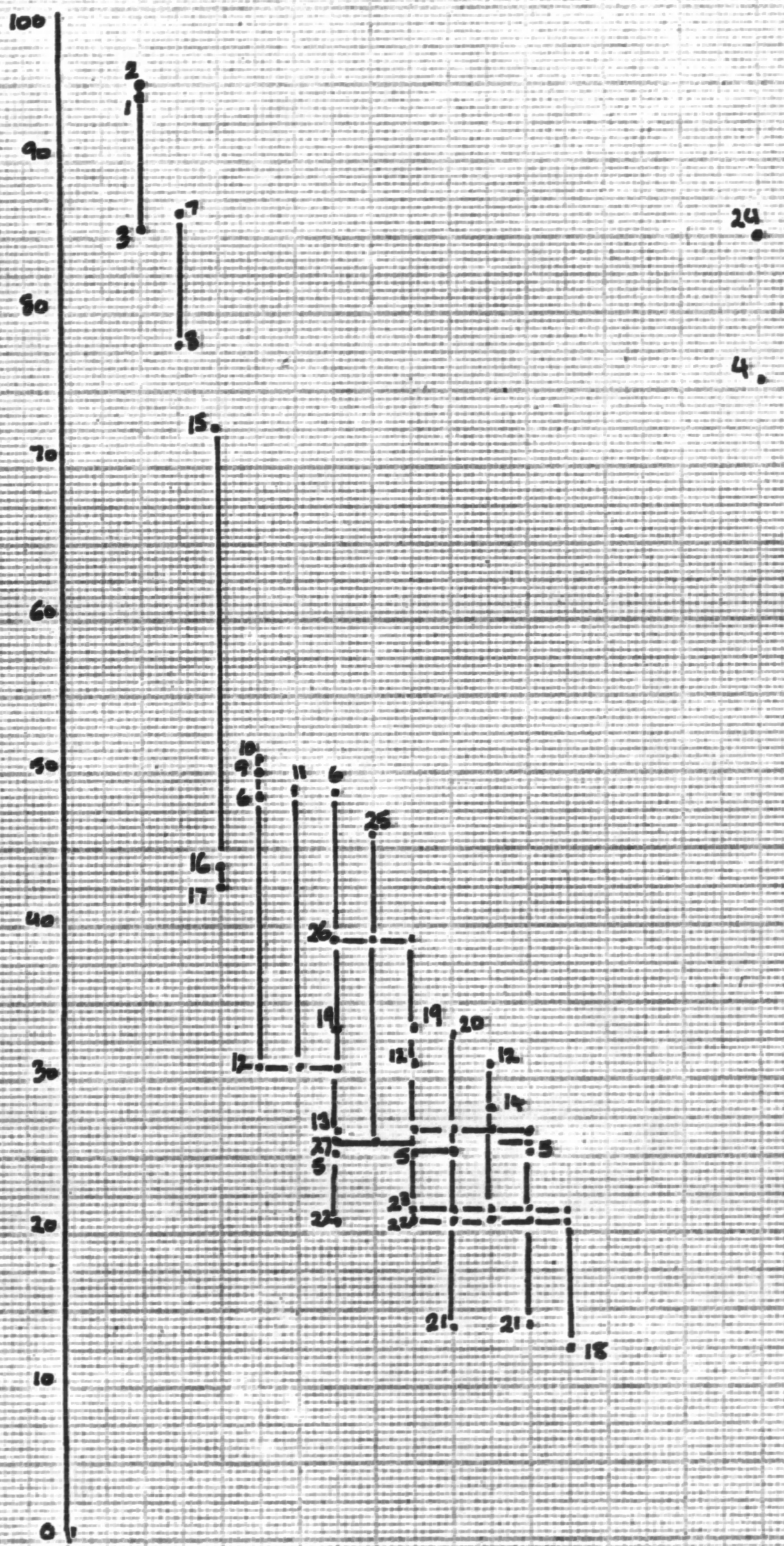


Fig. 22. Analysis of Data Using Complete Linkage $\phi \geq .35$



Items that appeared in more than one were joined, where possible, by a line parallel to the x axis, the chains being signified by lines parallel to the facility axis. Items which did not enter any chain were omitted.

With a criterion value of 0.4 many chains were formed but each contained only a few items and they often consisted of items which were different parts of the same question e.g. variables 12, 13 and 14 which have a natural similarity. When the criterion was reduced to 0.35 the chains spanned wider facility ranges, the groups were larger and there was considerable overlap. (The final group shown here would not strictly have been formed by the procedure above but was included as variable 18 only just failed to reach the criterion with variable 21). Simplification was attempted by removing some of the items which did not appear to be performing well. Variables 15, 16 and 17 were parts of a geometric problem which, although highly interrelated (they required exactly the same operation), had very low correlations with all other items on the test, they formed strand C in the previous discussion. Variable 25 appeared in a small chain with the other percentage questions and so appeared more to test the ability to cope with the particular question rather than the general subject matter of the rest of the test. Variable 20 was also withdrawn at this stage because although it had moderate correlations with the more difficult items it had low correlations with the slightly easier ones and therefore did not fit the general pattern. Variable 11 was retained since it only just failed to meet the criterion level with the other four items in the same facility range.

Both methods of plotting items which were connected by a ϕ of a specified level reduced the number of items being considered. Items which had few such linkages were ignored. A start had been made therefore on grouping items, the cut-off between successive groups was howeger still undecided. If a large facility gap was apparent this was used as a cut-off point, thus variables 1, 2, 3, 7, 8 seemed likely candidates for a group.

The easy items posed a special problem because they fell into two groups which had low correlations between them. However, since there were few items at this level they were assigned to a single group, which was used as a criterion for starting the remainder of the test. This action was supported by the fact that all of these five items had high Loevinger H_{ij} coefficients with the harder items. Variables 19, 12, 14, 13, 27 and 5 seemed to form a reasonable group. Their facilities were all similar and intercorrelations were good (the correlations between variables 14 and 5 (.348) and between variables 14 and 19 (.332) were the only ones to fail to achieve this criterion). Similarly at an easier level variables 10, 9 and 6 combined with variable 26 somewhere in between, variable 11 was included because of its correlation with all the other members of this group (.32, .334, .323, .345). Variable 26 was included with the easier of the two groups because its highest correlation was with variable 6. Variables 23, 22, 21 and 18 also formed a reasonably homogeneous group in the greatest difficulty range (the only low correlation - between variables 21 and 18 was 0.322), although the separation of variables 22 and 23 from the harder items in the group above was based on other criteria. These criteria were:

a) Mathematical coherence

b) Looking at the performance of the children in terms of a particular error; variables 18, 22 and 23 were often solved incorrectly because of the use of the addition strategy.

Groups of just two items were amalgamated with others since two seemed to be too few to be describable in any general sense. To test the scalability of the groups arrived at, each child in the sample was assigned to a level corresponding to the most difficult group in which he correctly solved two thirds of the items, then a Guttman scalogram analysis was carried out. If a preponderance of errors appeared between two particular adjacent groups, these groups were further investigated to see whether the cut between them could be changed without denying any of the criteria already described.

A pass mark of two thirds was an arbitrary decision, a half seemed too low for the statement "has at least achieved this level" and all correct seemed too strict a demand. The Guttman scalogram analysis was likely to result in no gross percentage error since items randomly solved by the children had already been rejected on the basis of Ø.

Decisions had to be taken throughout this analysis, on an acceptable level of ϕ, on the criteria for the inclusion of items within a group and cut-off points for the groups. Other mathematics educators took part in discussions of both the rationale for these decisions and the coherence of the items in both mathematical demand and methods used by the children. Other researchers using the same data might have arrived at different groupings since the delineation of one set of items as a group is by no means unique. The foregoing analysis produced a hierarchy of groups of items, the pass mark assigned to each group and the ϕ values between items in the groups is shown in table 29 for both 1976 and 1977 data.

Table 29. Definition of Levels Within the Hierarchy of Understanding in Ratio and Proportion

		Mean ϕ	Median ϕ
		1976 (77)	1976 (77)
Level 0	Less than 3/5 of level 1		
Level 1	3/5 of variables 1, 2, 3, 7, 8	.31 (.348)	.17 (.25)
Level 2	3/5 of variables 10, 9, 11, 6, 26	.39 (.33)	.35 (.315)
Level 3	4/6 of variables 19, 12, 14, 5, 27, 13	.41 (.39)	.37 (.36)
Level 4	3/4 of variables 23, 22, 21, 18	.41 (.34)	.38 (.305)

1976 data.

Children giving error responses 3.6 percent (n=2257)

Coefficient of Reproducibility .9816

Coefficient of Scalability .9127

1977 data.

Children giving error responses 3.36 percent (n=743)

Coefficient of Reproducibility .9832

Coefficient of Scalability .9123

The items omitted were all three parts of the question dealing with the doubling of the line segments in an open figure; variable 24 because it was essentially the introduction to a question; variable 4 which was part of the recipe question; variable 20 which was the first part of the chemical compounds question and variable 25 which was the first percentage question. The data from the 1977 sample (n=743) was subjected to the same type of analysis. Variables 15, 16, 17 had very low correlations with every item but each other and so were rejected as they had been when the 1976 data were used, variable 24 was not considered. The values of Ø obtained from the 1977 data might have led one to include variable 25 in group 2, it was one of the percentage questions and was quite highly correlated (Ø =.42) with another percentage question, variable 26. Variable 18, which was again the hardest question on the paper, had lower Ø values with other items in group 4 than in the 1976 data and might very well have been omitted. Generally the groupings would have been the same if the 1977 data had been analysed first.

The association between items within a group has been shown statistically; the mathematical demand they have in common was brought to bear when the delineation of one set from another was under consideration. The groups of items were given to a number of mathematics educators involved in research and also to practising teachers. The methods used by children in the interviews were also described to them. After discussion the levels were described in terms of the most naive strategies used by children for their correct solution. The description of the levels appears in table 30.

Table 30. Description of Levels in Hierarchy for Ratio and Proportion

Level	Description	Items
0	Unable to make a coherent attempt at any of the level 1 questions	Less than 3/5 on level 1
1	No rate needed or rate given. Multiplication by 2, 3 or taking half.	3/5 correct of items 1a(i) 1a(ii), 1b(i), 3a(i), 3a(ii)
2	Rate easy to find or answer can be obtained by taking an amount then half as much again	3/5 correct of items 2, 3a(iii),3a(iv), 3b(i),8c
3	Rate must be found and is harder to find than above. Fraction operation also in this group	4/6 correct of items 3b(ii) 3b(iii), 3b(iv), 1b(iii) 5, 8d.
4	Must recognise that ratio is needed, the questions are complex in either numbers needed or setting.	3/4 correct of items 4b, 6b, 7a, 7b

Using the pass marks already described the percentage of children in each year group who achieved each level is shown in table 31.

Table 31. Percentage of Children at Each Level (1976 and 1977)
(The 1977 figures are shown in brackets)

	2nd year	3rd year	4th year	
	n=800 (296)	n=767 (257)	n=690 (190)	
Level 0	7 (11.8)	7 (7.4)	3 (4.7)	percent
Level 1	53 (54.1)	49 (44.7)	41 (40.5)	
Level 2	26 (19.3)	23 (26.1)	27 (27.9)	
Level 3	9 (8.4)	12 (13.2)	14 (14.2)	
Level 4	5 (3)	9 (7)	15 (6.8)	

The majority of children did not solve the items which appear in levels higher than level 2. There is also little difference between the percentage of children at each lefel in each year. Further evidence of this slow development in the understanding of ratio and proportion is provided by the results of the longitudinal survey in which children who were second years in 1976 were tested on the paper in 1976, 77 and 78.

Results of the Longitudinal Survey

The sample chosen for the longitudinal survey appears on p105, the survey covered two years, the children being tested in the summers of 1976, 77 and 78. By the end of two years 99 children had attempted the test paper three times, all the others had been absent on one or more occasions when the paper was given. This diminution in numbers meant that the IQ ranges were represented as follows:

IQ $\leqslant$ 89	n=21 from 4 schools
90 $\leqslant$ IQ $\leqslant$ 99	n=28 from 4 schools
100 $\leqslant$ IQ $\leqslant$ 109	n=23 from 4 schools
110 $\leqslant$ IQ	n=27 from 4 schools

The children were assigned to levels on the basis of their performance on the ratio test each year and a comparison was made between their scores on the three occasions on which they attempted the test. Figures 23-27 show the progression year by year. The progression was slow and not composed of a great leap in any one year. In addition the performances at the beginning and end of the two year period were very closely allied to IQ. For example, no child in the group with IQ less than 89 was performing at level 4 either at the beginning or end of the three testings and no child with IQ greater than 110 was at level 0. The details of progress within each IQ group were as follows:

$IQ \leq 89$. No child moved more than one level in any one year. Over two years two children moved two levels. Nine children stayed at the same level over the entire period.

$90 \leq IQ \leq 99$ In any one year only one child moved two levels, three others moved two levels in two years and nine stayed at the same level over the testing period

$100 \leq IQ \leq 110$ Two children moved two levels in any one year. One other moved two levels in two years. A number regressed in some way and are discussed below. Four children stayed at the same level over the two years.

$110 \leq IQ$ Four children moved more than one level in any one year, two others moved more than one level over the entire period. Five children stayed at the same level (one of these was at level 4 in 1976).

The results of the large scale testing, which showed there was little difference in performance between the year groups, are supported by the findings from the longitudinal survey. An increase of one level over two years was the most common form of progress.

Some children appeared to regress (regressions are shown in black on figures 23-27) and others were "error types" in that they achieved a pass mark on a group of items without passing all easier groups. The regressions could be split into three types: 1) a regression at the second testing with a subsequent improvement 2) a regression at one level (i.e. less items in a group correct) with an increase in score on other levels 3) true regression, where the child's score in 1978 was worse than in 1976. Table 32 lists the children who regressed in one of these three ways. The three scores (1976, 77 and 78) on each group of items is shown.

Fig. 23 Longitudinal Study - Ratio Levels (Total Sample n=99)

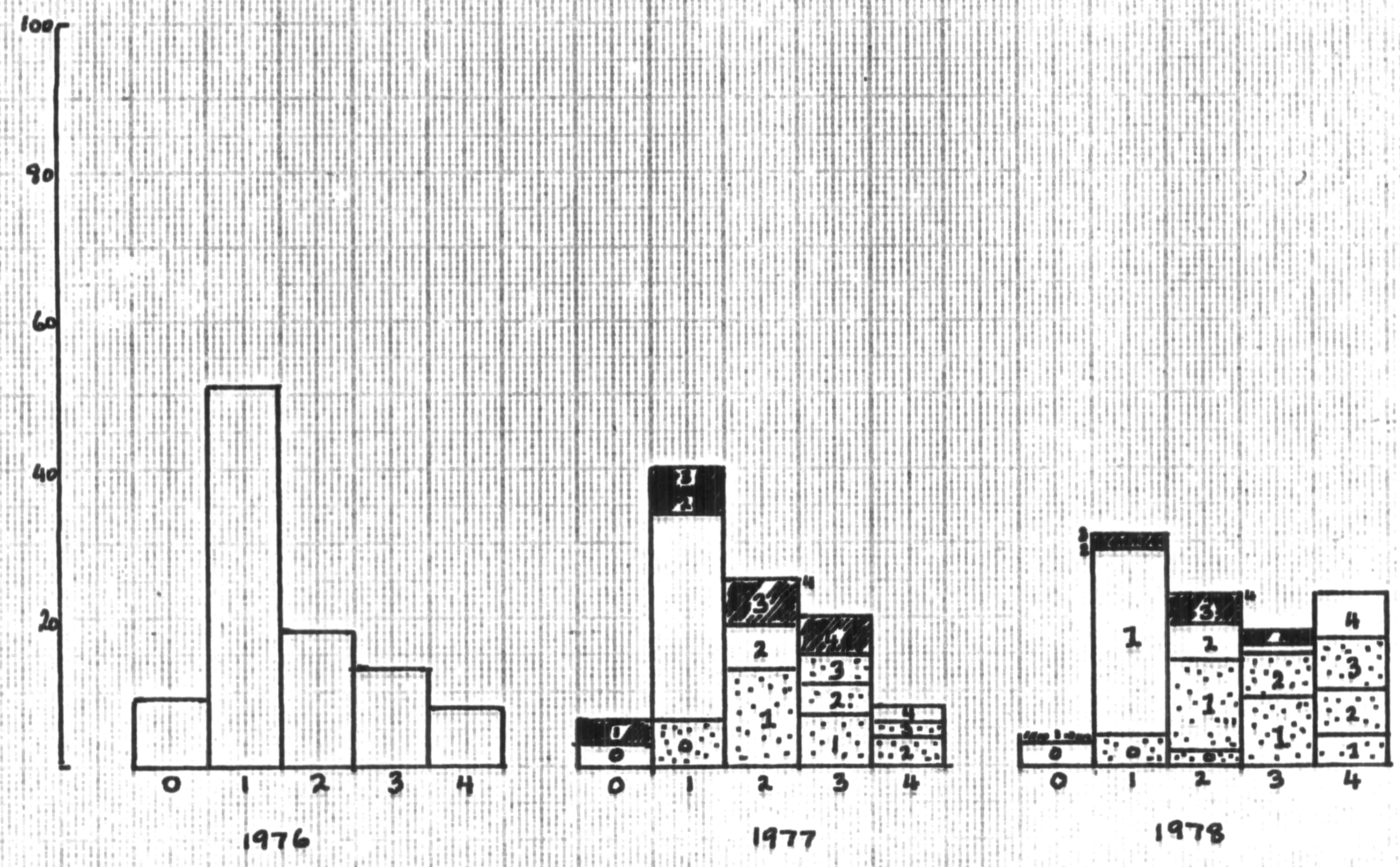


Fig. 24 Longitudinal Study - Ratio Levels IQ $\leq$89 (n=21)

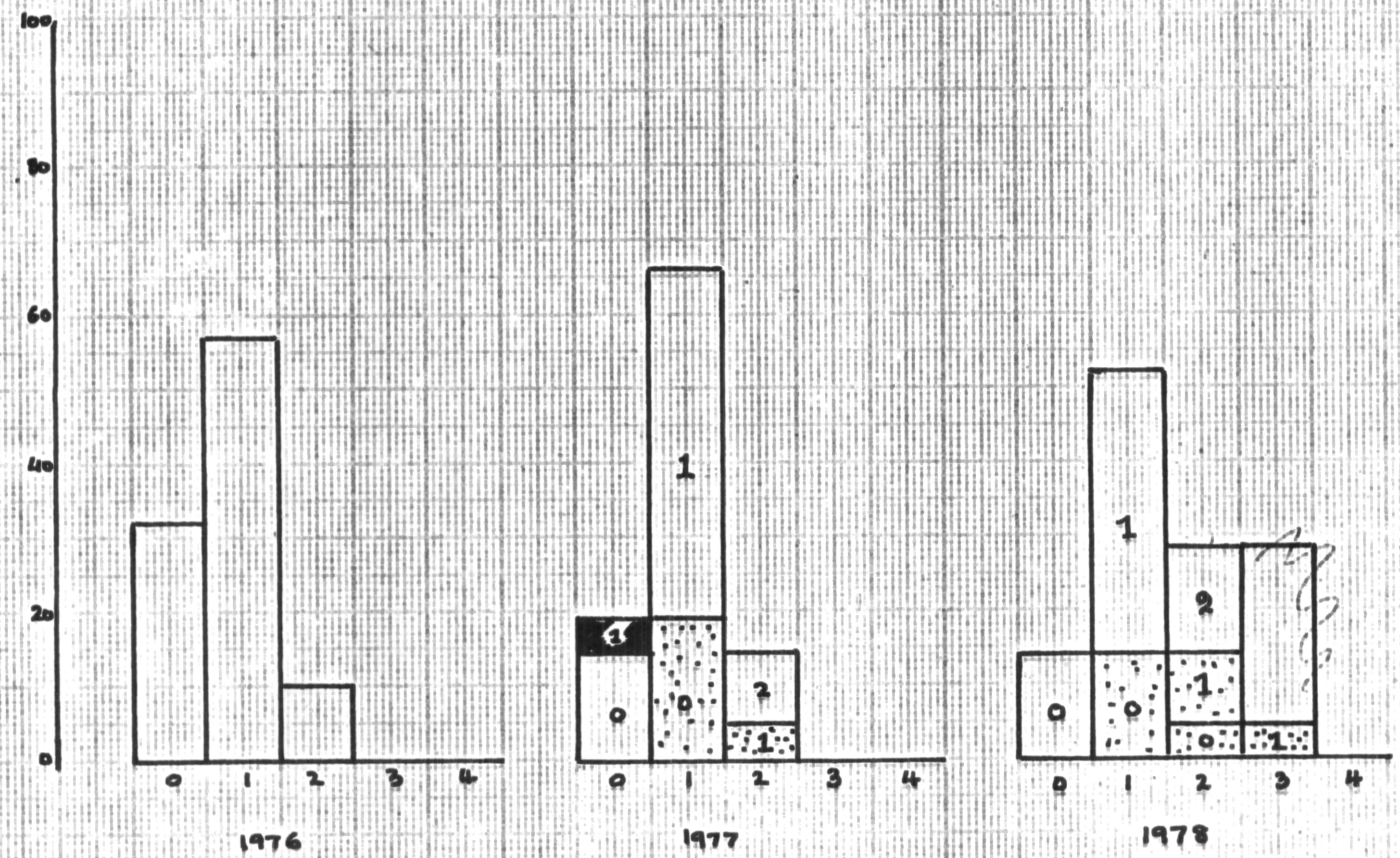


Fig. 25 Longitudinal Study - Ratio Levels 90 ≤ IQ ≤ 99 (n=28)

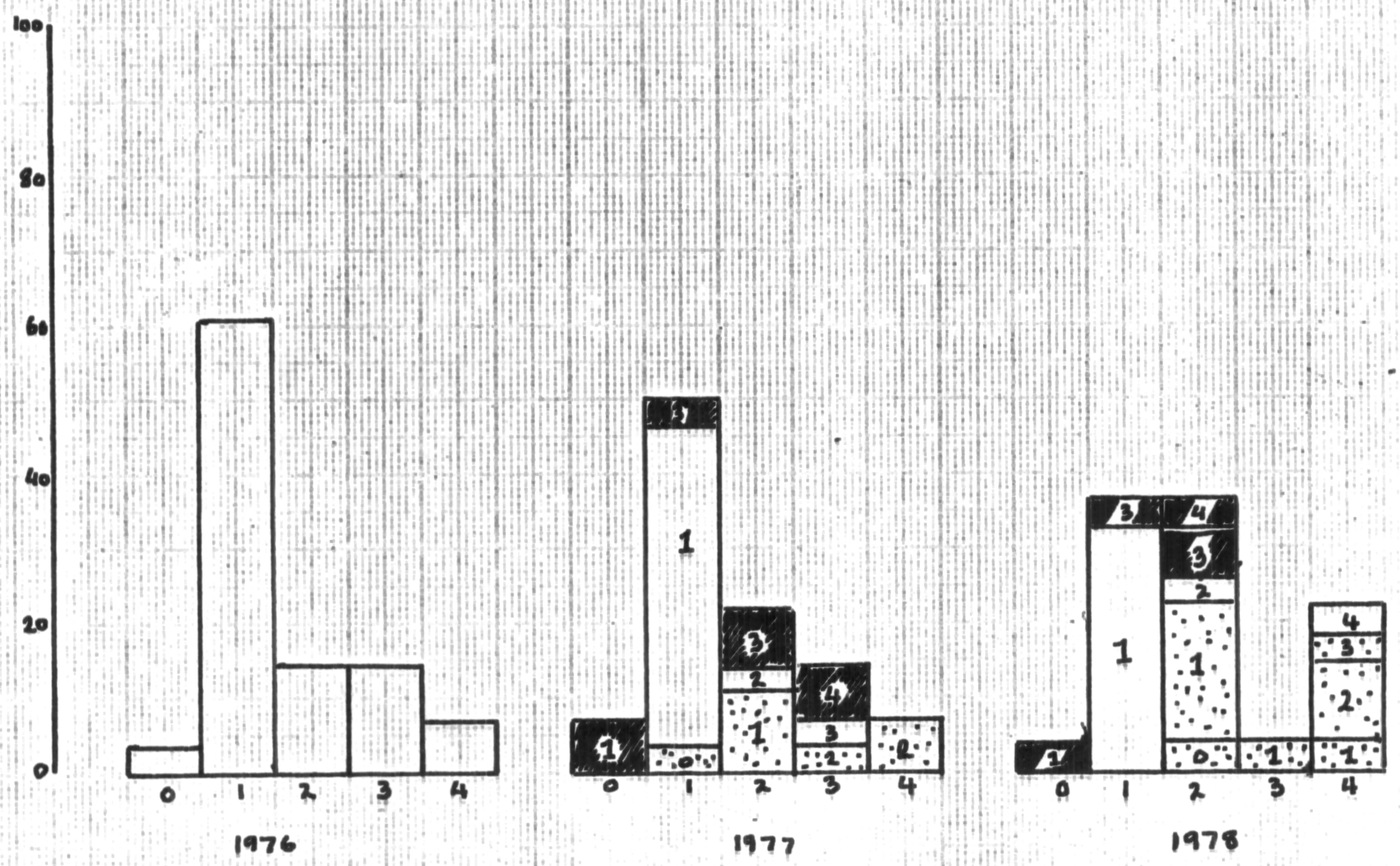


Fig. 26 Longitudinal Study - Ratio Levels $100 \leq IQ \leq 109$ (n=23)

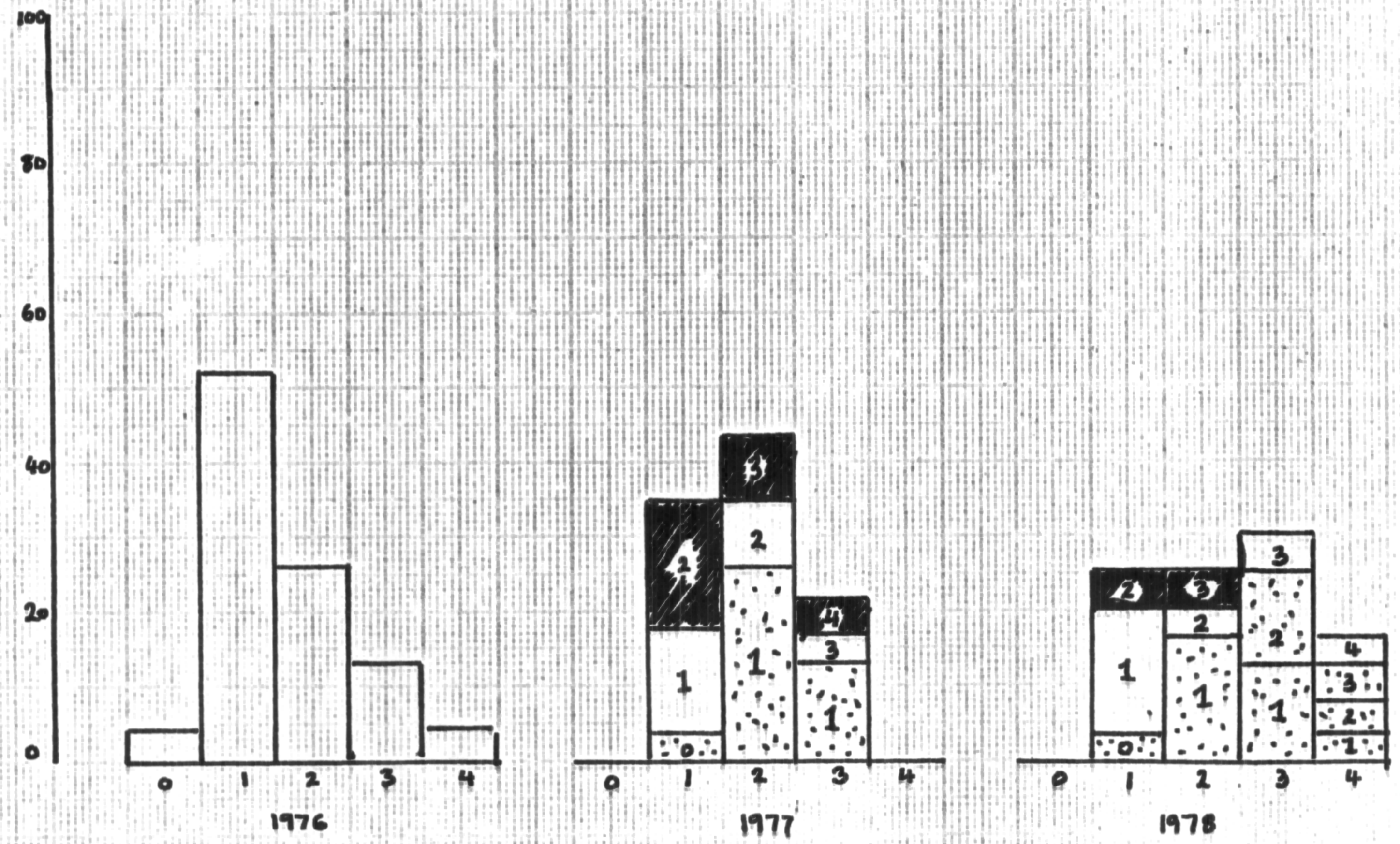


Fig. 27 Longitudinal Study - Ratio Levels IQ $\geq$ 110 (n=27)

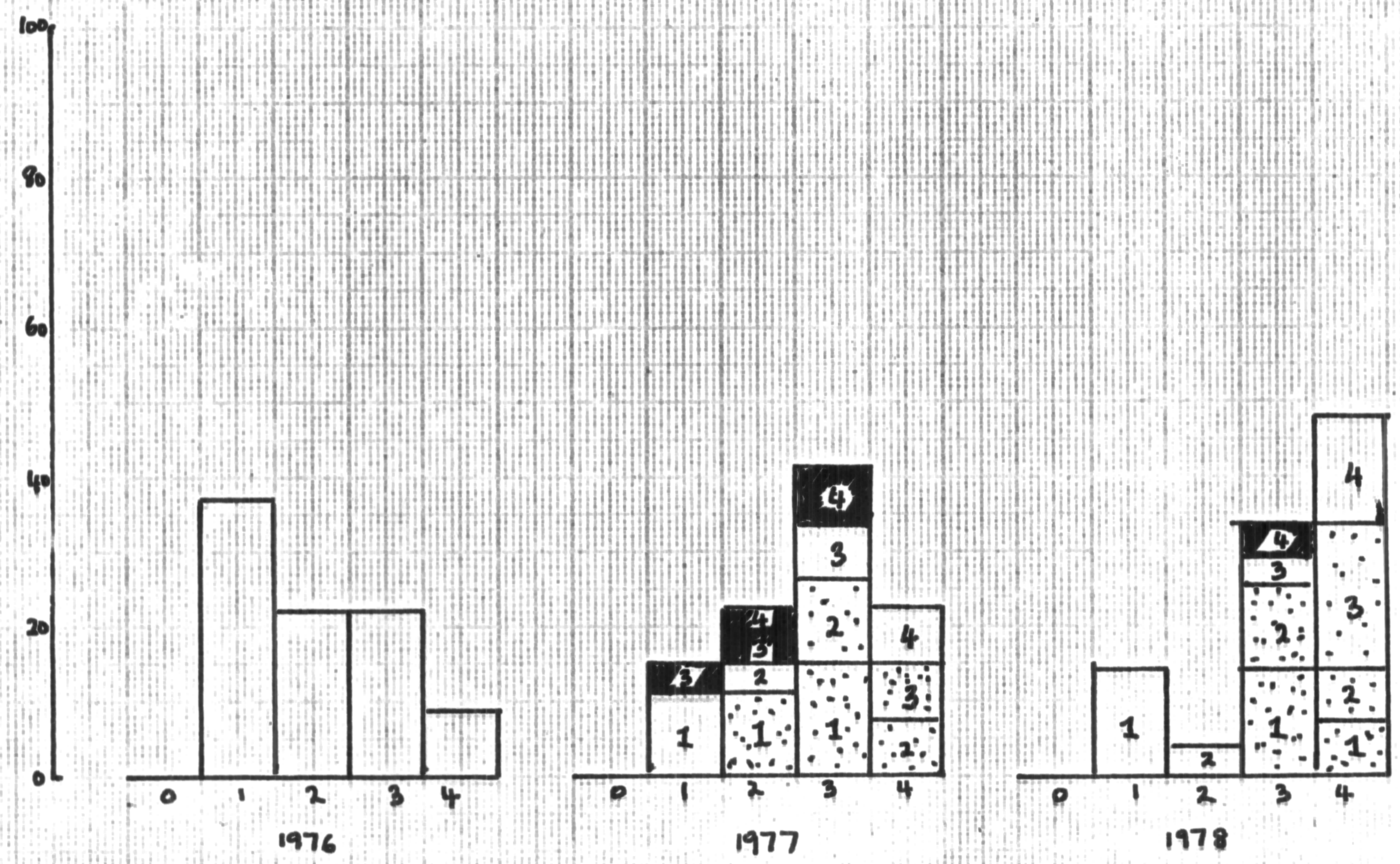


TABLE 32 Regressions in Longitudinal Survey

Regression Type	Child Identifying Number	Scores on Levels (1976,77,78)			
		Level 1	Level 2	Level 3	Level 4
1	011529	304	111	001	000
	(IQ $\leq$89)				
	071550	555	345	465	303
	071626	325	014	021	001
	(90 $\leq$ IQ $\leq$ 99)				
	011508	554	313	111	000
	071502	545	445	535	114
	161509	555	424	004	001
	161649	555	324	114	000
	(100 $\leq$ IQ $\leq$109)				
	011506	554	545	432	003
	071527	555	545	556	423
	071634	555	554	536	303
	161583	555	445	456	423
	161598	535	322	425	102
	(IQ $\geq$110)				
2	011509	545	313	432	001
	(90 $\leq$ IQ $\leq$99)				
	071504	555	435	613	101
	(100 $\leq$ IQ $\leq$109)				
	161842	555	141	003	000
	(IQ $\geq$110)				
3	071526	556	344	521	121
	(90 $\leq$ IQ $\leq$99)				
	071592	455	255	263	022
	161610	555	322	201	000
	051673	555	151	022	000
	(100 $\leq$ IQ $\leq$110)				

Note Type 1 - Regression 2nd year, subsequent improvement
Type 2 - Regression at 1 level, increase score other levels
Type 3 - True Regression

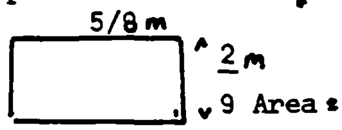
The true regressions were few in number and the other two types appeared overwhelmingly in the higher IQ groups (IQ $\geqslant$ 100). Only one child in the IQ $\leqslant$ 89 band regressed in any way over the three testings. Four children regressed from a level which they had achieved without passing all easier groups and which could be regarded therefore as a spurious attainment. There is considerable movement between levels in the third year in Secondary School but after this most children tend to regain the level they held before, or improve.

Comparison With Other Areas in Mathematics

The C.S.M.S. mathematics team investigated nine other topics which commonly appear in the Secondary School mathematics curriculum. It is beyond the scope of this thesis to go into the details of the hierarchies in each of the other topics. A general description of the type of item which corresponds in facility to level 1 Ratio items would involve conventions and terminology. The base line for all the tests was a knowledge of whole numbers up to a thousand and the ability to use numbers up to 20. Items comparable to those in level 1 Ratio, demand one extra dimension of complexity, e.g. comparing whole numbers or the meaning of 1/2 or collecting letters (not numbers) in Algebra. Levels 2 and 3 Ratio items correspond in facility to items on other papers which require the use of a strategy for solution or which require the first degree of abstraction e.g. Vector questions are not necessarily accompanied by diagrams and in Graphs the child is asked to interpret a graph and not simply read off the position of a point. Level 4 Ratio questions correspond to those in other topics which require abstractions and the appreciation of mathematics as an abstract system. To solve items at this level the child must have moved away from the confines of the set of whole numbers and realise the potential of decimals and fractions e.g. he must see that there is an answer to 16 $\div$ 20 and not simply say "it is impossible" as he did when working with whole numbers.

During the interviews carried out prior to the wide scale testing it soon became apparent that children avoided using fractions, they seldom multiplied by a fraction and when a fraction occurred in the questions, as in the recipe, the facility of the item dropped markedly. C.S.M.S. tested children on two fraction test papers, one involving only addition, subtraction and meaning of a fraction, for the 1st and 2nd years in the secondary school and another which required multiplication and division as well, for the 3rd and 4th years. Each paper was in two parts, a set of problems and then a set of computations; the computations mirrored the problems. A comparison of the hierarchies obtained from the results of the ratio paper and the fraction problem paper for the 3rd and 4th years appears as table 33. The gamma coefficient (Horst 1966) which quantifies the probability that if child A is ranked higher than child B on one test, he is also ranked higher than B on a second test was found.

TABLE 33. Hierarchies in Ratio and Fractions

Ratio n = 2257			Fractions 3, 4 n = 523
Doubling, halving No rate i.e. amount per person not needed	Level 1	Level 1	Labelling part of a whole Equivalent fractions where doubling is involved 1/3=2/? Addition same demominator
75-80% Facility			
		Level 2	Equivalent fractions not of obvious doubling form e.g. "A relay race is run in stages of 1/8 Km each. Each runner runs one stage. How many runners would be required to run a total distance of 3/4Km?
52-58% Facility			
Obtains answer by repetition,	Level 2		
		Level 3	Double complexity of equivalence e.g. 2/7=4/14=10/?
Operation on a fraction e.g. 1/2(1/2)+1/2(1/2 of 1/2) Converse of repetition (diminution instead of enlargement) or rate must be found and then repetition take place.	Level 3		
22-30% Facility			
Ratio is viewed as a multiplicative not additive operation, ratio itself is 3:2 or 5:3	Level 4	Level 4	Multiplication of fractions e.g. 
Error rate 3.7%			Error rate 2.9%

Note $\gamma = .85$

It can be seen that the ability to halve, which was required in level 1 Ratio is comparable to equivalence when doubling is needed and that the multiplication of fractions occurs at the same facility level as the use of a fractional multiplier in ratio. Every child in the C.S.M.S. sample attempted two test papers and 68 children in the 14-15 age group attempted both the Fraction and Ratio test papers. A matching of their performance on each of the papers is shown below.

Fig 28. Crosstabulation of Performance on Fraction Problems and Ratio

		Fractions Levels 0	1	2	3	4	Row Total
Ratio Levels	0	4	1	0	0	0	5
	1	6	11	10	4	1	32
	2	0	1	7	12	2	22
	3	0	0	0	6	1	7
	4	0	0	0	1	1	2
Column Total		10	13	17	23	5	68

Note. Number of children appear as cell entries.

It can be seen that a performance at two extremes did not occur but that a legel 1 Ratio performance could be matched by levels 1 and 2 in Fractions, recall that there was no Ratio level parallel in facility to level 2 Fractions. The same children were also given a series of fractional computations. The comparison between ratio facilities and the computations were as shown in Fig. 29. The gamma coefficient was . 6197, i.e. considerably lower than when the comparison was between the same Ratio levels and the levels obtained from groups of the Fraction problems. This tends to confirm the findings of Abramowitz (1975) who stated "....skill tests of facility with fractions load on a different factor than tasks involving proportionality".

Many children found the fraction problems easier than the corresponding computations, giving support to the possible hypothesis that they may not employ taught algorithms when they have available their own successful (even if naive) methods.

FIG.29 COMPARISON OF HIERARCHIES IN RATIO AND FRACTION COMPUTATION

Facility	Ratio		Fractional computation	
100				
		95		
			78	$3 \times 10\frac{1}{2}$
80	Level 1			
		73		
			68	$1 - 5/12$
60				
		57	54	$1/3 + 1/4$
	Level 2			
			42	$1\frac{1}{5} - \frac{3}{5}$
40		41		
		32	30	
	Level 3			$3/4 \div 1/8$
		23	26	
20		20	22	$4\frac{1}{2} \div 12$
	Level 4			
		6	13	$3\frac{3}{4} \times 2\frac{1}{2}$

RATIO

FRACTIONAL COMPUTATION

The Addition Strategy

In chapter 5 certain children were designated "adders" because they used the incorrect addition strategy identified by Karplus (1975) on three or four of the items 4b, 5, 7a and 7b. That is, they concentrated on the difference a-b rather than on the ratio a/b and simply added that difference in order to enlarge. Their performance on individual items has been described in that chapter. Their performance in terms of levels of understanding on the ratio paper is as follows (note by definition they fail on level 4 items since they use the addition strategy on two of the four items at that level):

Table 34. Levels of Understanding (Adders 1976)

Year	No. Adders	Levels	0	1	2	3
2	246		6	138	78	24
3	220		5	138	57	20
4	156		2	83	59	12

The Karplus question (item 5) appeared at level 3 so those children who achieved level 3 must have succeeded on the other items or not used the addition strategy on this particular question. Table 34 shows that the large majority of adders appear to be at level 1 although a sizeable number can successfully deal with level 2 items.

The longitudinal study shows that children are consistent in their use of the incorrect addition strategy on items 4b, 5, 7a and 7b. Of those in the study at all three testing periods, 37 were using the addition strategy in 1976 and of these 16 were still using it in 1978. Six children who were designated "adders" in 1976 successfully solved three of the four items in 1978, the other 15 children had two or more of the four items incorrect in 1978 (although not necessarily by using the addition strategy).

The four questions on which the addition strategy appeared most frequently provided diagrams. Three of them involved similar figures and the fourth was the Mr. Short and Mr. Tall problem, all of them concerned comparison of sizes rather than sharing quantities 'so that it would be fair' as in the recipe or eel questions. It could be hypothesised that the actual provision of the diagrams prompted children to use the addition strategy and that the same type of problem without the diagram would be attempted in a different way. Consequently a small study was undertaken involving two classes in two schools. The questions on the original test paper were given to the children and in addition questions which used the same ratio and the same numbers as items 5, 7a and 7b but which did not provide diagrams were included; these were:

10 Susan has two dolls Snoopy and Sally. When she measures Snoopy's height using a clothes peg, he is six clothes pegs tall. When she measures Snoopy's height using a pencil, he is four pencils tall. Sally, the other doll is six pencils tall.

How many clothes pegs tall is Sally? ....

11 A photographer has a picture of the side view of a chair 

a) He wants to enlarge the picture. In the small picture the length of a chair leg is 8 units, in the larger picture he wants a chair leg to be 12 units.
If the seat width is 9 units in the small picture, what will the seat width be in the large picture?
....

b) If the chair back is 18 units in the large picture what would it have been in the small picture?
....

The total number of children tested was 55, all third years. The purpose of the rewritten items was to see whether the children who used the addition strategy on questions 5, 7a and 7b on the original paper would continue to do so when the questions were in a different form. The consistency of the method is shown in table 35 below.

Table 35. Comparison Study - Addition Strategy

Original Item	New Item	
5 (20 adders)	10	Four children added on item 5 and had item 10 correct. Three children had item 5 correct and added on 10. Thirteen children added on both.
7a (16 adders)	11a	One child added on 7a and had 11a correct Two children had 7a correct and added on 11a Eight children added on both
7b (12 adders)	11b	Two children added on 7b and had 11b correct One child had 7b correct and added on 11b. Six children added on both.

It seems likely therefore that the use of the addition strategy is very much more concerned with the type of question i.e. enlargement of figures, than with the actual provision of diagram.

A second small study in which adults were given the ratio paper was also carried out in order to see whether those who were to teach children understood ratio and whether adulthood necessarily entailed understanding.

Performance of Students in Colleges of Education

A number of adults, mainly students in Colleges of Education, took the ratio test. No attempt was made to obtain a representative sample of adults and the following results are given simply to illustrate the fact that adults (of higher academic performance than the rest of the population) find the topic of ratio and proportion difficult.

Table 36. Performance on the Ratio Test (Adults)

College	No.	Level 0	1	2	3	4		Adders
A	23	0	0	9	17	74	percent	0
B	17	0	0	18	12	70		6
C	47	0	11	17	40	32		13
D	17	0	24	29	12	35		18
E	16	0	25	25	50	0		37.5
F	30	0	0	3	20	7		0

The students at colleges A and B were experienced teachers who were attending a course of retraining as Science teachers in the case of college B and mathematics teachers in the case of college A. The rest of the students were in colleges of education training to be teachers, although not specialist teachers of mathematics.

Piagetian Tasks

At each stage of this study an attempt has been made to match the demands of the ratio questions to cognitive levels as defined by Piaget. Certain items (3a, 3b and 4b) on the ratio test were taken from the researches of Piaget. At the pilot stage of the study, reported in chapter 3, the child's performance on the written test was matched to his performance on these Piagetian items and in addition on an extra item taken from Epistemologie de la Fonction (1968). The additional item appears in Appendix 4 . No clear assessment of the cognitive level of the child could be made on the basis of performance on this set of items since there was considerable lack of consistency. It was thought possible that a child's cognitive level could be assessed if he was given a series of mathematical tasks taken from the works of Piaget. A class test based on tasks taken from the works of the Piagetian school was written. A number of examples used in Geneva in clinical interviews were adapted into a class test format, pilot tested and then five tasks were decided upon because they presented less problems when given to children in written form. These five tasks appear in Appendix 16. The tasks were given to some 420 children, aged 13+ to 15+ in the summer of 1978. The research team of C.S.M.S. presented the paper and introduced each question, explaining what it required. The children were allowed to ask questions if they did not understand and they were encouraged to show (on the papers) their methods of solving the problems. Each child then did one of the C.S.M.S. mathematics tests (Ratio, Algebra, Graphs or Fractions). As far as possible the tasks were marked according to the descriptions in the books from which they were taken, on occasion the methods of the children did not correspond to any of the descriptions recorded by the researchers in Geneva. In this latter case the methods were assigned to a stage, after considerable discussion by the C.S.M.S. mathematics team based on descriptions of performance in other works by Piaget and on his general descriptions of the cognitive levels.

It was hoped that a clear cognitive level could be assigned to each child on the basis of his overall performance on the five tasks. This level would then be compared with the child's performance on a mathematics test. A consistent pattern Piaget level/mathematics level would have enabled each stage in the hierarchy to have been described in terms of the Piagetian cognitive stages. This proved to be unrealistic since the children did not perform consistently on the Piagetian tasks and the type of answer they gave (late concrete etc.) depended to a very large extent on the particular task set. The γ Coefficients obtained from comparing performance task/task on the Piagetian test and task/mathematucs test are shown in tables 37 and 38.

TABLE 37. GAMMA COEFFICIENT ITEM/ITEMS ON THE PIAGET TEST

Q1					
Q2	314 .269				
Q3	420 .373	394 .370			
Q4	375 .447	358 .250	449 .451		
Q5	358 .474	327 .325	419 .394	368 .407	
542	Q1	Q2	Q3	Q4	Q5

TABLE 38. GAMMA COEFFICIENT PIAGET TASKS/MATHEMATICS LEVELS

Q1	70 .658	289 .316	79 .417	128 .230
Q2	78 .541	260 .364	94 .299	80 .261
Q3	89 .655	354 .530	98 .658	151 .571
Q4	82 .684	329 .479	98 .454	140 .215
Q5	80 .596	287 .365	84 .493	120 .230
	Algebra 89	Ratio 367	Graphs 102	Fractions 160

It can be seen that the γ coefficient between tasks on the Piaget paper were on the whole lower than those between individual Piaget tasks and the mathematics tests. The γ coefficients between mathematics tests were very much higher, many being around .8. The lack of consistency on the Piagetian tasks may have been due to drawbacks in the test paper or in the marking scheme. Piaget himself of course conducted clinical interviews and the lack of consistency may have been caused because the tasks were presented in paper and pencil format. A similar lack of consistency in performance on a range of Piagetian tasks has however been reported by Winkelman (1975). In the discussion of the implications for teaching in the next chapter, the question of Piagetian levels is again raised. No clear statement can be made from the Piagetian task paper as far as descriptions of the ratio levels is concerned.

CHAPTER SEVEN

Discussion and Implications of the Research

The research on the understanding of ratio and proportion reported in this thesis was based on the topic as it is taught in the secondary school and how it is understood by children of secondary school age. A large number of children were tested and the results of the written test paper were interpreted in the light of the interviews with children (which took place prior to the large scale testing). The information obtained therefore is based on what is thought to be representative of English children's understanding of the topic of ratio and proportion, bearing in mind certain limitations of the research.

In order to test a large number of children, pencil and paper tasks were used, these by their very nature give less information about a child's understanding than does a clinical interview or continued observation. The levels in the hierarchy are levels of attainment on a test <u>but</u> it is suggested that since these levels are scalable, homogeneous and described in terms of child interviews they approximate more to levels of understanding than would ranking acoording to total score. A second limitation of the study is concerned with the choice of sample. The sample is chosen from schools which volunteered to help. The teachers in these schools were attending in - service courses when they offerred their help and did not feel threatened by an assessment taking place in their classes. It might be assumed therefore that they would be among the more confident of mathematics teachers. An attempt was made to include both rural and urban schools within the sample but no account was taken of the socio-economic background of the pupils, size of school or the curriculum used in the school.

The ratio test paper was given to the children during a normal mathematics lesson by the normal mathematics teacher. Instructions for the administration of the test were given to the teachers but individuals may have interpreted them in different ways. It has been assumed that the conditions under which the children took the test made little difference to the overall success rate on the items.

The test paper is to be published by the National Foundation for Educational Research together with a teachers' guide describing the marking scheme , the demand of items and the hierarchy. How a teacher might interpret the marking codes given to a child i.e. what type of errors he is making , is also included. An example of these marking instructions appears in Appendix 17.

The hierarchy of levels of understanding based on groups of items at different facilities shows that the easiest application of ratio was enlargement by 2 :1 or 3 :1; 1:1 was not tested. The items at level one implied a rate, so much per person or so much per eel length or sharing so that it was 'fair'. It is conceivable that in addition to the numbers in the ratio, the understanding of 'fairness' when amounts were distributed contributed to the high facility. The fact that 2:1 or 1/2 is very specific knowledge and readily available to the child has already been mentioned by Karplus (1972b) and Piaget (1976). The analysis of the test paper on Fractions mentioned on page 197 also showed that this specificity occurs in other contexts; the fraction 'one half' being more easily recognised and its equivalents more easily found than for other fractions. It is of course more often used in everyday life both in measurement and in money than any other fraction, indeed few other fractions are used at all. When compared with level one questions in other topics the use of 2:1 was comparable in difficulty to the knowledge of conventions used in fractions and decimals i.e the recognition of the meaning of parts and also to those items which required a single step or operation on two elements.

Level 2 questions were at about 50 per cent facility, (level 1 questions varied from 75 to 95 percent facility), there was a large gap between the 2: 1 questions and the type which appeared in level 2. The three eel questions at this level involved 3:2 or 5:2 but we know from the interviews that the most popular 'child method' for doing these was an extension of doubling and halving; the children tended to say "take it once take a half and add". The easiest percentage question occurred at this level- "24 out of 800" written as a percentage. The lighting question, although asking for amounts obtained from a threefold ratio 3:2:1 could be completed by taking half and then half again since the amount paid by two men together was equal to the amount paid by the third. Neither level one or two truly required the manipulation of fractions, on interview such manipulation was avoided as long as possible. The question on cream which appeared in the recipe, the other parts of which were easiest on the paper, is a case in point; if the building up method was used the ensuing computation involved $\frac{1}{2}$ of $\frac{1}{2}$ add $\frac{1}{2}$ of ($\frac{1}{2}$ of $\frac{1}{2}$) which defeated many children. Twenty percent in fact opted for the answer one third, which seemed reasonable and did not involve adding fractions. The cream question was the hardest of the level 3 items. The other level 3 items were from the second eel questions which not only involved the use of continuous quantities which involved the distractor of the same unit of measurement for both the fish and the food but also had ratios other than 2:1, 3:1 and 3:2. The building up method could be used but the child had to invent another step not explicit in the question before he could use it e.g. he had to find the amount for the smallest eel even though this was not asked for. Two of the questions gave the amount for the largest eel and so the building up had to become a scaling down and as one child said "its a minus isn't it?". The question taken from Karplus (1974) also occurred in level 3, this required the ratio 3:2 but was subject to the plausibility of the incorrect addition strategy since the correct answer was not very different from the one obtained by concentrating on the difference a-b rather than on the ratio a:b. Also in this question there occurred the repetition of a number i.e. $a/b = b/c$, which Abramowitz (1975) has already stated is an added complication. Thirty percent of the third years tested (age 14+) successfully dealt with this item, Karplus (1974) reported that of his sample in the same age range (from seven different countries) 25 percent correctly answered the question. The final item in level 3 was the percentage question dealing with the reduction in the price of a coat, this required not only the computation of five percent of 20 but also a subtraction so was of a degree of complexity rather greater than the percentage question which appeared in level 2. The items in level 3 therefore were either subject to distractors or added a further dimension to the application of the methods which appeared to be used in the level 2 items.

Three of the items in level 4 were concerned with similar figures, two being non rectilinear with the ratio of enlargement of 3:2 and the third being the enlargement of the open rectilinear figure  in the ratio 5:3. The fourth item involved the ratio 3:2 but this was obtained indirectly by using a:c and b:c in order to find a:b. The addition strategy was used by very nearly half the population on two of the similarity questions. The multiplicative aspect of enlargement is by no means obvious to children; in the case of doubling, the enlargement proved to be easy but this could be carried out by the repetition of the line segment or a correct addition method. The building up method could be used in the computation of an enlargement of 5:3 but just as in the case of the cream question the resulting addition involved fractions.

Performance of the Children

The children investigated in this study were from three age groups in the secondary school (second to fourth year). It was apparent from the results that a difference in age was not necessarily indicative of a difference in performance on the ratio items. There were children at every age level who could deal with the level four questions just as there were children at each age level who were restricted to the level 1 type question only. Indeed the results from the colleges of education showed that gaining adulthood did not necessarily show an absolute awareness of the nature of ratio and proportion (this was in fact the point made by Renner, 1977). The questions which required ratios other than 2:1 and 3:1 had facilities of fifty percent or less, so half the population was very limited in its use of the concept of ratio. Those who might be regarded as able to cope with complexities involved in the solution of problems requiring ratio or proportion were those able to deal with level 4 questions, these formed less than 20 percent of each year group. For the successful solution of items of this type the child must be able to handle fractions either by using them directly to enlarge or by dealing with them when they arise in a building up method of solution.

The results of the longitudinal study showed that although there was a progression from year to year it was seldom of more than one level; the vast majority of children moved one level or stayed at the same level they had achieved in their second year in school. The child performance appeared to be closely linked to the IQ of the individual in that the different IQ groups investigated in the longitudinal survey showed markedly different performances. The results for the fourth year sample were rather better than those for the second year sample just as the children performed a little better when they were aged fifteen than when they were aged twelve, the difference was insufficient however to support any hypothesis that children of fifteen are automatically able to deal with proportion. Some children, the minority, appear to be fairly flexible in their use of ratio no matter what their age (in this study restricted to secondary ages).

The children who were interviewed showed considerable lack of consistency in the correct methods they used, as was described in chapter 5 with reference to the eel question. Even the brightest children (who could deal with multiplication of fractions) did not in fact use this method on questions where a more naive method would suffice. A preference for the operation of addition was always in evidence and confirmed the findings of Lunzer (1966) who said with reference to his own research which used examples different to those quoted in this study:

> What the above researches suggest is that at all ages children seem to prefer to look for additive modes of solution even when the problem could suggest multiplicative methods. When the former are of no avail, success is not reached until well into the secondary years. p.11.

The results from the testing with the ratio paper although confirming the desire on the part of the child to add gave no clear indication that an increase in age automatically led to an increase in the willingness to use multiplication. A further confirmation of the preference for addition was supplied by the existence of thirty percent of the sample who when faced with problems where multiplication was needed opted for the incorrect addition strategy and chose to consider the difference a-b rather than a:b and added this difference in order to enlarge. The building up method where small segments of the answer are obtained and then added does not appear as a method of solving ratio and proportion problems, in mathematics text books currently used in British schools. It is however a method used by many people when faced with an everyday problem which requires enlargement, just as many people do not use subtraction (the method taught in schools) but 'counting on' when they are checking their change after a purchase. The methods suggested by textbook writers are 1) enlargement by a scale factor and the use of a centre of enlargement (a geometric construction) when diagrams are to be made bigger, 2) the rule of three or cross-multiplication in the equation a/b = c/d when three values are known and one is not or 3) the unitary method where a rate is found by reducing the question to "how much for one?". The geometric construction aspect was not tested on the ratio paper since it was considered to be a skill rather than an indication of the understanding of the meaning of enlargement. One child on interview attempted to use this method but when applied to the items on the test paper it proved to be very difficult. Indeed the rectilinear figures were so drawn that the centres of enlargement were by no means obvious. The cross multiplication method was not used at all by the children interviewed and since it has to be written down in some form it was relatively easy to identify when used on the written test. Only twenty children out of the sample of 2257 tested in 1976 did in fact write it down and use it successfully. Of these, fifteen were from the same school. There were a hundred children tested in the school and this fifteen (from three different classes) were on the whole the children who did best on the paper. The conclusion to be drawn is that the others had been taught the method but did not use it, possibly because they did not see its relevence or because they were unable to recall it when the need arose. One child on interview said "I must find out how much for one" but none wrote the three sentences which usually accompany the teaching of the unitary method viz:

5 yards of calico cost 65p

1 yard costs 65 ÷ 5

8 yards costs 8 x $\frac{65p}{5}$

There seemed to be little evidence of the use of a taught algorithm being used on the questions which appeared on the ratio paper. Those children who succeeded on the harder questions appeared to have adapted the algorithm into multiplication by a fraction and those who succeeded only on the easier questions had adapted whatever they were taught into a building up method. The method of acquiring segments of a solution and then adding at the end occurred many times in the interviews even though the children were from a number of different schools.

The incorrect addition strategy was used by children from all schools and indeed as Karplus (1974) has pointed out, from schools in many different countries. The items which attracted it most were those which required an enlargement of a figure and not a sharing among people. Even when the figure enlargement was described in words rather than diagrammatically, the addition strategy was very evident (see pp 201). It is possible that children see the enlargement of diagrams as being essentially additive and if they are not shown the result of such addition and the distortions to which it inevitably leads, they are content to have provided a larger figure.

The addition strategy was not only used consistently on the four difficult questions but also persistently year after year as shown by the children in the longitudinal survey (pp 200). Of those who added when they first attempted the paper 50 percent were still using this method when they tried the paper for the third time, even though their teachers had been told of their adherence to this incorrect method.

<u>Cognitive Levels</u>

As explained in chapter 6 the class test composed of tasks taken from the works of Piaget did not prove to be an adequate method of ascertaining the cognitive demand of the comparable level on the ratio test. Certain items on the ratio paper were taken from the rewearch of Piaget on ratio and proportion i.e. the first eel question and item 4b. There was very little evidence of Piagetian stages II and III on the eel question but if one accepts that the correct answers to question 3a are indicative of stage IV B then level 2 on the ratio paper approximates to stage IV B (age 9+ or late concrete). Question 4b appeared in level 4 of the ratio Hierarchy and was the hardest item at this level, its successful completion is regarded by Piaget as occurring at stage IV (age 11, early formal) when the ratio is 3| 6 .One could therefore postulate that level 4 demonstrates formal thinking, only the last question (about 20 percent of the fourth year solved it) requiring late formal thinking. An additional support for this statement is that the children who added (incorrectly) on level 4 items were at level 3 or below (mainly levels 2 and 1). Karplus stated that the addition strategy did not occur on a continuum leading to the successful application of the proportion schema. Piaget however stated that the child who concentrated on the difference a-b rather than on a/b was demonstrating late concrete thinking. If one accepts this diagnosis then level 4 items are beyond the ability of these children and therefore demand more than late concrete thinking;

The task on the Piagetian test which had the highest γ coefficient with ratio was in itself a question on proportional reasoning in which the child was required to enlarge a rectangle (see appendix question 3). The gamma coefficient was .530. Table 39 below shows the crosstabulation ratio levels/Piagetian stages on task 3; it can be seen that the majority of children were giving late concrete answers (2B or 2B-3A) to the enlargement of a rectangle question. No clear assignment of ratio level to Piagetian stages is possible although children at below level 3 in ratio appear to be below early formal reasoning on the rectangle question.

Although no suggestion is made that the levels form an ordinal scale there is evidence to support the idea that the demand of level 4 items is greater than the demand of levels 2 and 3. There are far fewer children solving level 4 items and it has been shown that success on the harder group entails success on all easier groups of items. It is only at the last stage of the hierarchy that the child is not distracted by the incorrect addition strategy and resolves , in terms of Inhelder (1974) the conflict between what appears plausible and what is correct. The recognition at this stage that ratio is needed could be regarded as the emergence of problem solving strategies rather than the application of a 'method'. In Blooks taxonomy applications and problem solving are higher order manifestations of understanding.

Attainment on the ratio test is closely linked to the IQ score of the child, see the longitudinal study, and the values of the gamma coeffient between ratio and the other mathematical tests devised by CSMS point to the fact that the test itself was tapping rather more than rote learned rules and performance without understanding.

The methods used by the children on interview showed that teacher taught rules were adapted by the child and used in a form different from that which was presented in class. If these methods were to be of use to the child they had to be internalised (assimilation and accomodation in Piagetian terms). The absence of algorithmic arguments in the inteeviews indicates that unless the algorithms can be seen to be relevant, they do not become part of the child's repertoire. Indeed the prevalence of additive strategies (both correct and incorrect) demonstrates that the child builds on that in which he has confidence i.e his ability to add. The theories of Ausubel (1971) formalise what is already fairly common practice in teaching i.e going from the known to the unknown or basing what is taught firmly on what the child already knows. It seems likely however in the teaching of ratio that the teacher overestimates the knowledge base of the child and tries to build on a confidence and knowledge of multiplication whereas the base is the operation of addition.

The difference between the performance of children of different ages is small and it seems inappropriate to postulate a course in Ratio tied to the age of the child. In the longitudinal study very few children regressed over the two years , what had been achieved appears to have been real and not the result of short term memory or rules without under.standing.

TABLE 39. Cross tabulation of Ratio Performance and Piagetian Task 3

TASK 3	COUNT ROW PCT COL PCT TOT PCT	RATIO Levels 0	1	2	3	4	ROW TOTAL
2A (early concrete)	2	2 25.0 10.5 .6	5 62.5 3.3 1.4	1 12.5 1.0 .3	0 0 0 0	0 0 0 0	8 2.3
2A-2B (early/late concrete)	3	2 5.9 10.5 .6	22 64.7 14.7 6.2	6 17.6 6.1 1.7	4 11.8 8.3 1.1	0 0 0 0	34 9.6
2B (late concrete)	4	11 8.5 57.9 3.1	68 52.7 45.3 19.2	35 27.1 35.4 9.9	10 7.8 20.8 2.8	5 3.9 13.2 1.4	129 36.4
2B-3A (late concrete/early formal)	5	3 2.2 15.8 .8	51 37.2 34.0 14.4	50 36.5 50.5 14.1	22 16.1 45.8 6.2	11 8.0 28.9 3.1	137 38.7
3A (early formal)	6	1 10.0 5.3 .3	0 0 0 0	3 30.0 3.0 .8	3 30.0 6.3 .8	3 30.0 7.9 .8	10 2.8
3A-3B (early/late formal)	7	0 0 0 0	4 11.1 2.7 1.1	4 11.1 4.0 1.1	9 25.0 18.8 2.5	19 52.8 50.0 5.4	36 10.2
	COLUMN TOTAL	19 5.4	150 42.4	99 28.0	48 13.6	38 10.7	354 100.0

Implications for Teaching

The results of the testing show that there are children of every age at each level of understanding of ratio. The hypothesis that understanding improves with age is to a certain extent supported but the improvement is neither dramatic nor automatic; most children improve very slowly. There is some evidence from the longitudinal survey to show that children whilst acquiring the ability to deal with more difficult ratios lose the skill to deal with the easier. This is particularly true of children in the higher IQ ranges during their third year in secondary school. Those with IQ less than 100 tend not to be non scale types or display regressions, their progress is slow and systematic but of course they do not progress to level 4.

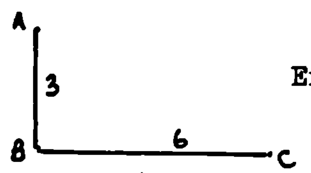
The lack of uniformity of performance in any one age group means that a teacher with a mixed ability class cannot expect all children to perform at the same level. The grasp of the topic varies considerably and therefore the material presented to the children also needs to be varied and to be chosen to match the particular level of the child. In order to match the mathematics to the level of the child the teacher needs to know which methods the child is using when he correctly solves a problem in ratio and proportion. The interviews already described, show that a child does not necessarily use the same method on all the questions but there appears to be an upper limit to the type of method available to him. For example if the recognition of the need for multiplication by a fraction and the successful use of it is regarded as the most sophisticated method used by children, then many will fail to apply this method no matter what the question requires. They will instead be limited to a building up method and find it very difficult to apply on more complicated ratio problems. Other children will be unable to use a building up method when ratios more complicated than 3:2 are involved. Unless the teacher is aware of the method the child uses he will be unable to understand both the errors committed and the lack of success on harder items. The commonly used method for correcting homework mistakes is for the teacher to demonstrate the correct use of an algorithm, without reference to the specific mistakes made by the pupils. This appears to be an unprofitable activity since the mistakes made by the children are varied and seldom involve an algorithm at all.

The teaching of a rule by which ratio and proportion questions can be solved poses its own problems. Teachers tend to introduce a rule and justify its use by presenting an easy example which can be solved by other methods. The results of this study show that very few children use the rule a/b = c/d even though taught it (see for example the fifteen children out of a hundred in one school who use it). It seems likely that while the questions can be solved by other means, the children will in fact use those other more accessible methods. By the time they are faced with examples which need more than a building up method and are therefore forced to consider a more sophisticated method they have forgotten what the rule was. There is also evidence that children who do not understand the rule do not remember it i.e. there is little support for the contention that children can apply rules blindly without understanding their significance.

The hierarchy of understanding in ratio and proportion shows that there is a very large gap between being able to apply the ratio 2:1 and being able to apply even 3:2 by a building up method (as in the eel question). Teachers should therefore be aware that being able to double is a poor indicator of the understanding of proportion. The slower children in a class, faced with a textbook exercise on ratio will often find that the first examples require 2:1, they may indeed never go beyond these examples. Some children in fact regard all enlargement as being a requirement to double and all fraction enlargement as a requirement to halve. The ability to interpret 5:3 is very far from the application of 2:1 and should not occur in the same exercise. A case could be made for introducing ratios with examples requiring say 5:3, thus forcing the children to appreciate the need for an algorithm. Since however this requires multiplication by a fraction and that in turn appears at level 4 on the fraction hierarchy (see pp197) the outcome is likely to be failure. Although it has been shown that the manipulation of fractions is not closely linked to either the solving of ratio or indeed fractional problems, the lack of ease with fractions as shown in the interviews is a barrier to even recognising when a ratio has to be applied. The problem in science lessons is greater since the results of experiments often result in non-integers being compared and if a child finds 5:3 difficult he is likely to be completely overwhelmed by 6.5/2.6. Multiplication of fractions is often introduced by recourse to the area of a rectangle as an illustration, followed almost immediately by the introduction of a rule. The CSMS results have shown that children often do not remember the formula for the area of a rectangle and if the dimensions are fractions such as 1/3 or 3/5, they declare that the question is impossible.

Many textbooks introduce the idea of enlargement by using similar figures usually triangles and rectangles. Both triangles and rectangles present problems when the child has to judge proportion. Gross distortions may convince a child that the two rectangles are not "the same shape" but the very words are ambiguous since rectangles are 'the same shape' in that they are rectangles. The words:- similar, the same shape, enlargement, all have other meanings in everyday language. "Enlargement" is probably the most technical but its usage in other than diagrammatic problems is awkward and 'negative enlargement' seems to be a contradiction. Many children do not appear to realise that an enlarged shape should look very much like the original (see the omission of the gaps in item 4a). It might be worthwhile to accompany exercises on enlargement by examples which show distortion or non-similarity. Except for the doubling of a line (item 4a), the three items on the test which involved similarity were all level 4 questions; the introduction of ratio and proportion through similar triangles, which is common practice, appears therefore to be open to question. The recipe questions were the easiest and in these the child was essentially asked "what is fair" or "how much per person". This aspect of proportion might be very much easier for the child to appreciate and should conceivably come first in teaching.

The IREM group in Orleans have been investigating proportion and in particular questioning which comparison between lengths the child makes. For example in this enlargement question does the child compare BC with a new base or AB with BC in the original diagram in order to find a ratio.



Enlarge this diagram

In fact saying enlarge BC by a factor and then multiply AB by the same number. The two children who compared BC with AB and were aware that the sole criterion for enlargement was that new base : new height should be in the ratio 2:1, were able to operate without stating an enlargement factor. One child in fact said "I didn't want to just double it. It might have seemed if I'd misunderstood the question if I'd just doubled it. Suppose it seems if I understood it better".

The results of the testing of a) the college of education students and b) the teachers being retrained as Science and Mathematics teachers showed that teachers themselves often do not appreciate the nature of ratio and proportion. Some are of course not intending to teach mathematics in the secondary school but all will be teaching children. In most colleges of education the course given to future primary teachers contains a small element of mathematics, in which the main emphasis is on the content of the primary school syllabus. Until 1980 when the requirement of 'O Level' mathematics for entry to colleges is introduced, many future primary school teachers will not be adept at mathematical thinking and will have had a career of failure in the subject. To insist that courses in mathematics should include the topic of ratio and proportion is to load an already overcrowded programme but the topic occurs so often in Science that one would think its importance was undisputed.

Future Research

This study has added to the literature on the child's understanding of ratio and proportion and has provided support for Lunzer's (1966) findings that children prefer to use an additive method for solution rather than a multiplicative one. The incorrect addition strategy seen as an extension of this correct addition was very common in the sample tested for this study, it is plausible and the child feels that he has a 'method' for doing the problems. Karplus and Kurtz (1977) have carried out research on successful teaching situations in which adolescents improved their understanding of ratio. The research showed that when children were placed in a laboratory setting with examples such as gears, where ratio was being employed, they often abandoned the addition strategy and their overall performance improved. Similar research is needed with younger children. Possibly the ratio aspects that are used in mathematics lessons are all too abstract and that it is only when the child sees the result of using ratio incorrectly on an object which has to move or perform in a specified way that he realises the importance of the concept.

The eradication of errors such as the addition strategy and doubling when not required need also to be investigated in the wider context of teaching programmes. Research is needed on what type of presentation by the teacher minimises these errors. We do not know which children recognise that their solution is wrong, certain children appear not to realise their mistakes and intervention by the teacher to show the gross distortions etc, which would result if the method was pursued, may be a way of bringing their attention to bear on the problem. Research on the effect of teacher intervention and the types of intervention which appear to be most fruitful is needed.

The hierarchy of course is based on the items which appeared in the ratio test paper, a similar study using different items but ones that possibly matched the general descriptions would provide additional information and validation for the hierarchy presented in this study. In the hierarchy there is a large gap between levels 1 and 2, this may occur because of the nature of ratio and proportion but it might be possible to find questions which differ from those in levels 1 and 2 but where the demand comes somewhere between the demand of these two levels. The hardest ratio question on the paper involves 5:3, it might be argued that once the child can solve this, more complicated questions are solved automatically. On the other hand the hierarchy could be extended to include examples of the type a:b:c and even the demands of trigonometry.

The sample used in this study came from English schools. The results might be indicative only of the content of the British school mathematics curriculum. A small study undertaken in Greece using the same test paper points to this not being the case, since the Greek children appear to perform in very much the same way and make the same type of mistake. Their scores on the whole were slightly lower than the British children of the same age, except on the hardest question (4b) where they did very much better. This is the question which uses 5:3. If Greek children manage to cope with this there is possibly a method of teaching being used in their schools which lessens the difficulty of the item. Information from other countries using the same test might point to other anomalies. Cross cultural studies on ratio would provide evidence of the generalisability of the ratio and proportion hierarchy.

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APPENDIX T

Items Used on Interview

CSMS

RATIO

(First Draft)

1. Tick which of the following are obviously true.
If not true state why.

a) Roger Bannister was the first man to run a mile in 4 minutes. He ran 5 miles in 20 minutes.

....

b) A Vienna loaf costs 12p, the shopper buys three, she pays 36p.

....

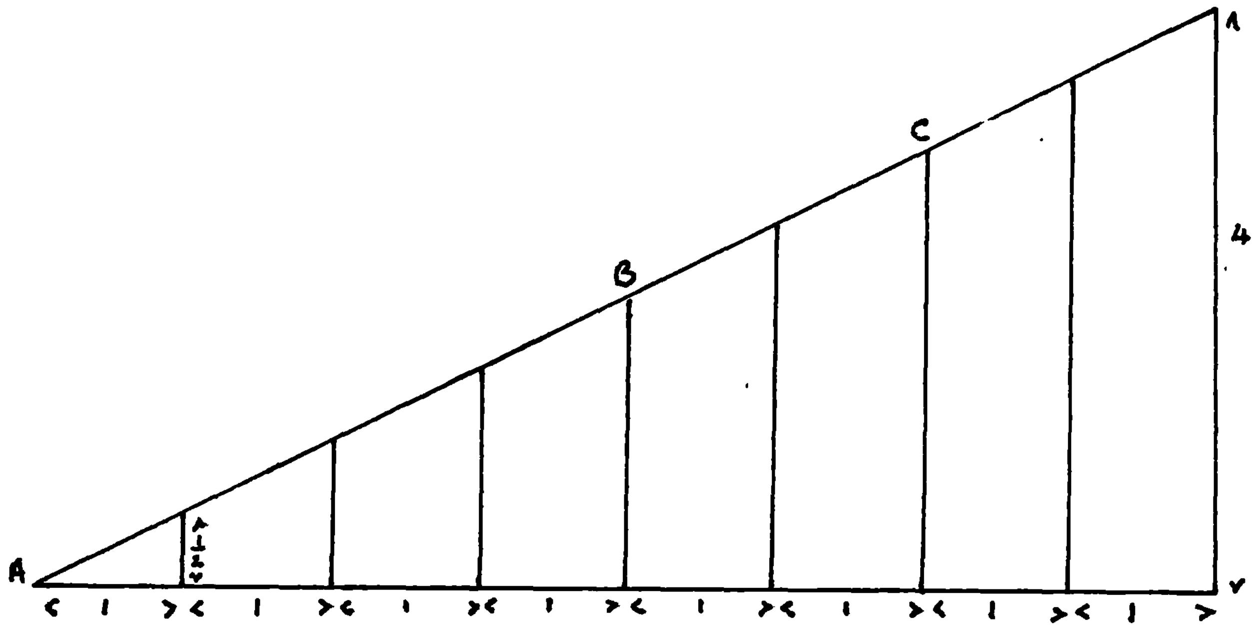
c) For one cake I use 10 ounces of flour, for two of these cakes I use 20 ounces.

....

d) A duster takes 40 minutes to dry, the three dusters I have hung on the line should take 2 hours to dry.

....

SIMILARITY



How high are the other uprights?

If I marked off 20 on the bottom line, how tall would be the upright before I hit the slant line?

If I marked off $4\frac{1}{2}$ on the bottom line, how tall would be the upright before I hit the slant line?

If AB is 4.5 which is AC likely to be?

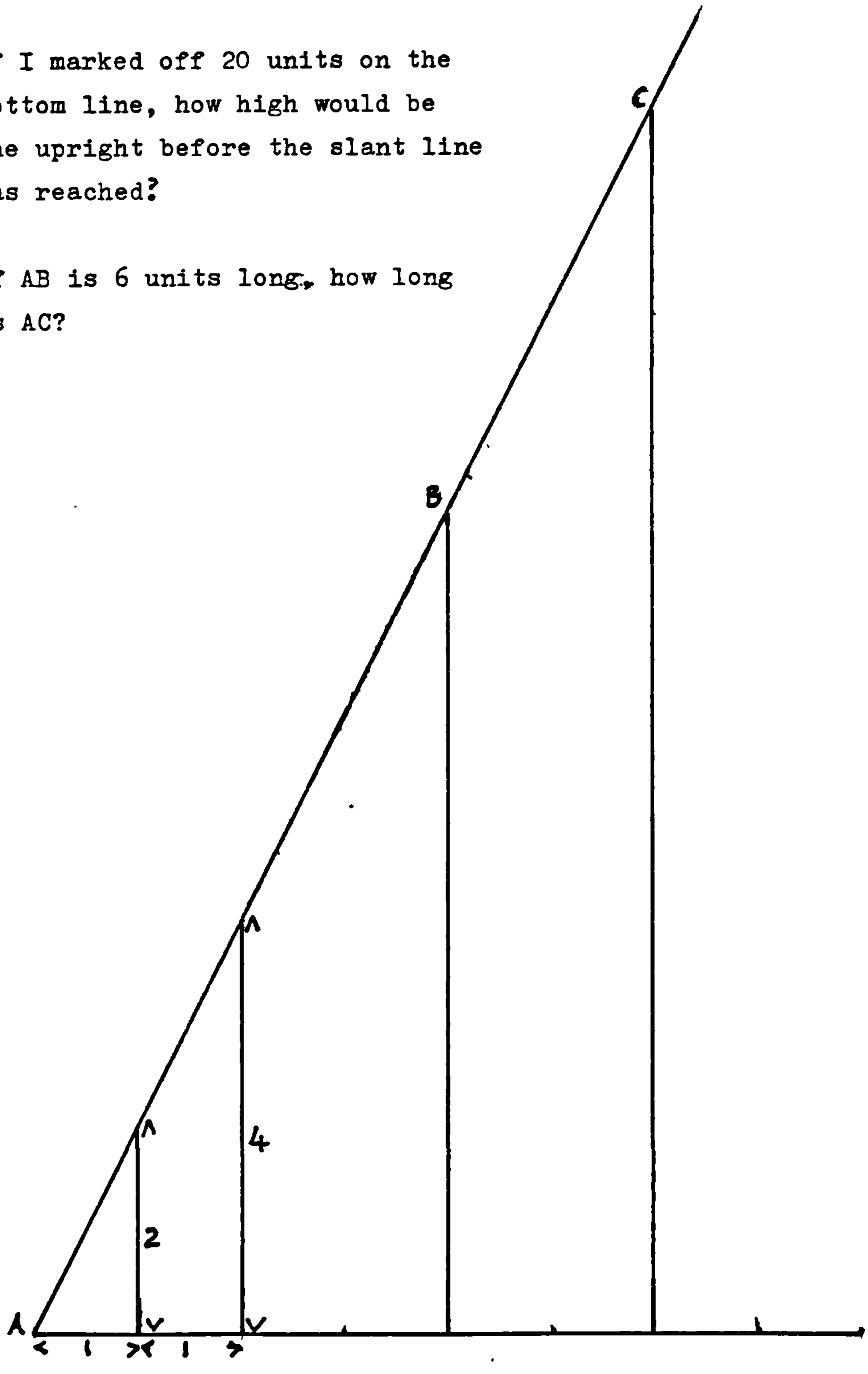
(a) $\frac{3}{2} \times 4.5$ (b) 3×4.5 (c) 6.5 (d) none of these.

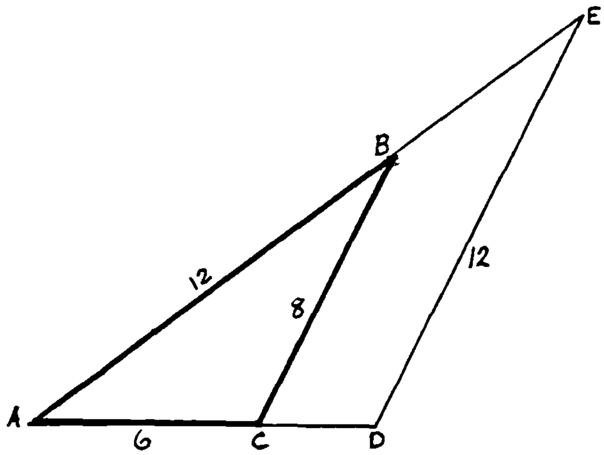
2

1) How tall are the two uprights shown?

2) If I marked off 20 units on the bottom line, how high would be the upright before the slant line was reached?

3) If AB is 6 units long, how long is AC?





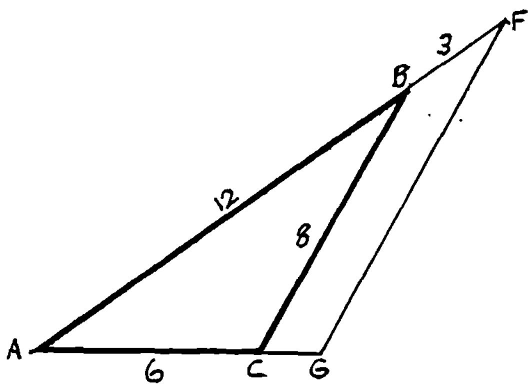
AB is 12 units

AC is 6 units

CB is 8 units

DE is 12 units

How long is AE? How long is AD?

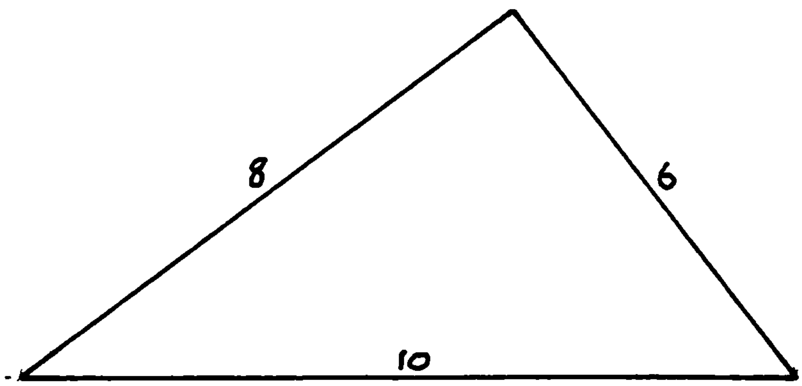


The triangle ABC is the same as above.

BF is 3 units

How long is FG? How long is AG?

4.



You are asked to make a triangle which is the same shape as this one but which can be larger or smaller. Imagine you are making a new triangle with meccano strips, you must start with the one marked 12. Which other strips would you use? Draw a rough diagram.

Are there any other possible triangles you could make, still starting with the one marked 12?

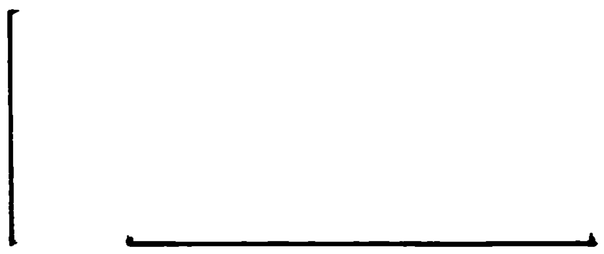
<u>Meccano strips (lengths)</u>

12 units   14 units
15 units
8 units
9 units
20 units
10 units
16 units

5 Enlarge the following diagram so that it is triple its size.

Finish drawing the diagram below so that it is the same shape but bigger than this diagram.

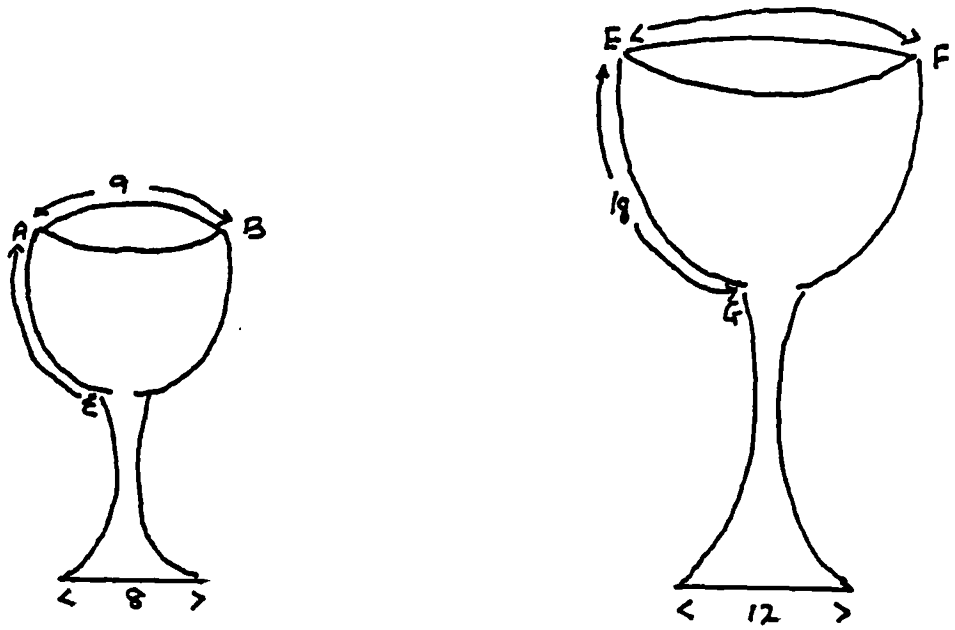
6 a) Enlarge this diagram, including the gap so that everything is double in size



b) We have enlarged one line of the original diagram. Put the other line similarly enlarged in its correct position so that you have an enlargement of all the original diagram.

Work out how long the missing line
should be if this diagram
is to be the same shape but bigger
than the one above. ....cm.

7

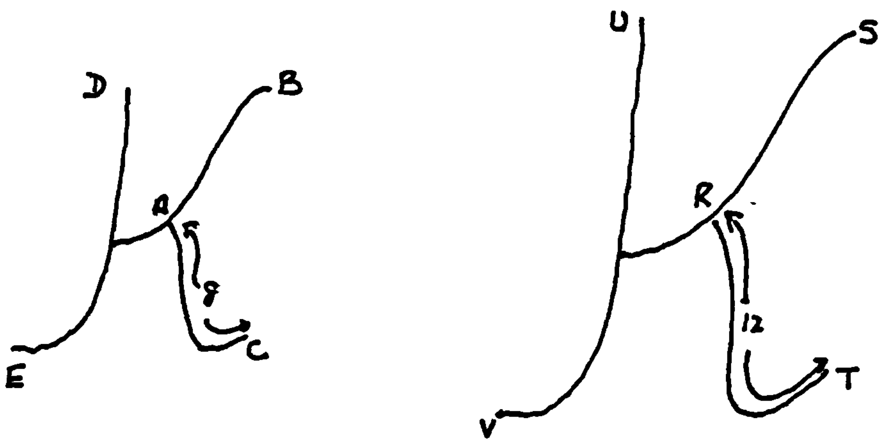


These two pictures are the same shape, one is bigger than the other.

The curve AB is 9 units. How long is the curve EF?

The curve EG is 18 units. How long is the curve AC?

8



These 2 letters are the same shape, one is larger than the other.

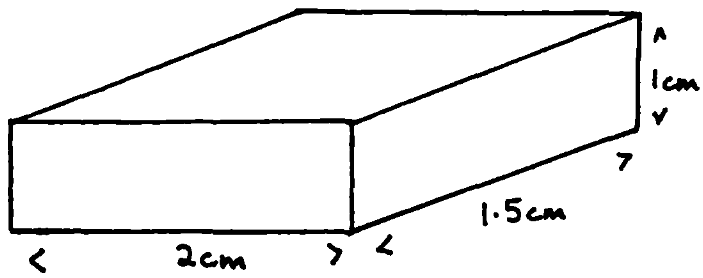
AC is 8 units. RT is 12 units.

The curve AB is 9 units. How long is the curve RS?

The curve UV is 18 units. How long is the curve DE?

Geometry - Scale Drawing

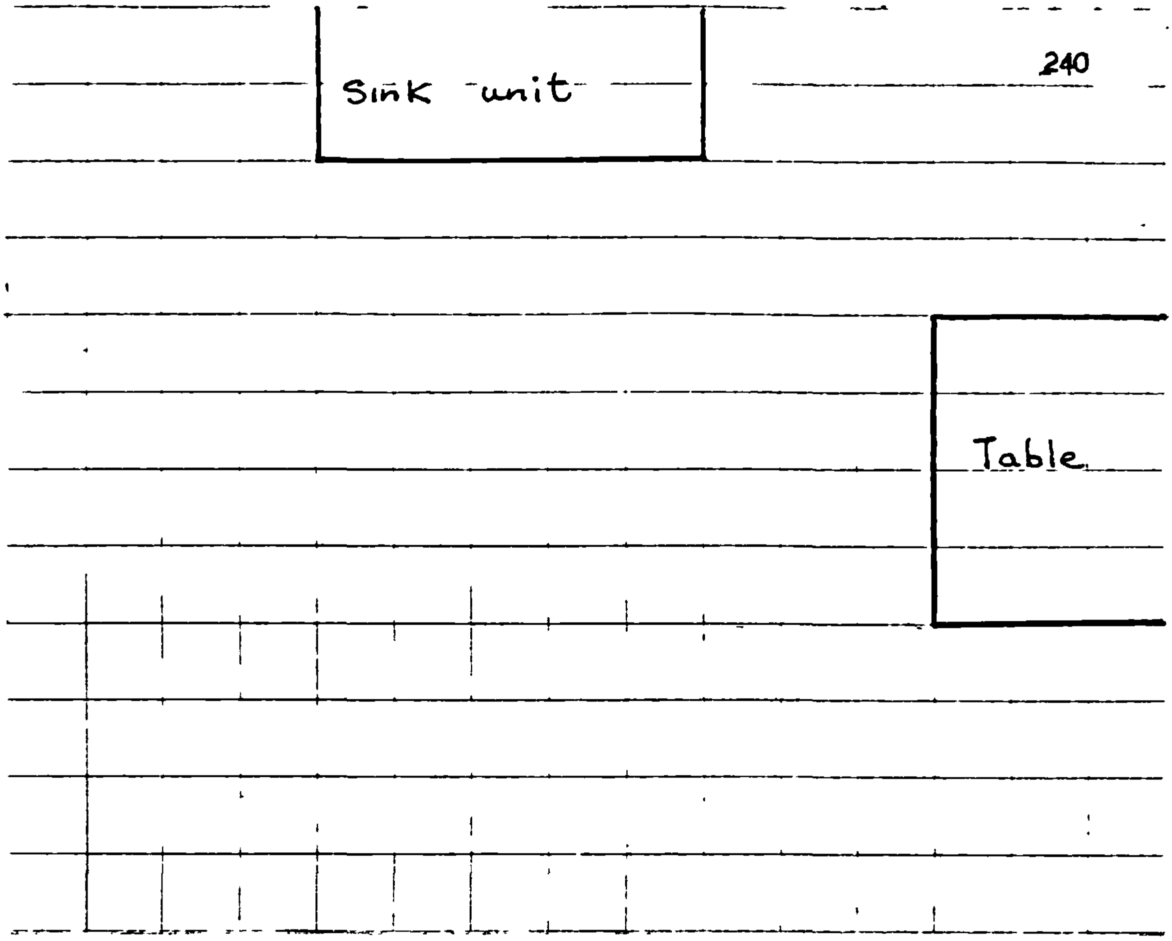
1.



The scale for the plan of the kitchen is 3cm to 2 metres. The refrigerator I intended putting into the kitchen is 1 metre wide. How much space should I allow for this on the plan? The window on the plan is 1.5cm above the ground, how high would a table have to be so that it exactly reached the bottom ledge of the window? How many cubic centimetres would the freezer hold if its dimensions on the plan were as above.

The kitchen itself is 4 metres by 3 metres. How big a sheet of paper do I need for the plan?

2.



This squared paper is marked in centimetres. It represents the floor of a room. We have marked the position of the sink unit which is 2.5 metres long. What scale is being used?

What are the dimensions of the table, a) on the plan; b) actually?

a) .... b) ....

I have a gas stove which has a base measuring 1 metre by 0.5 metres. Draw this on the plan and label it.

The fridge is 0.75 metres by 1 metre base. Draw this on the plan and label it.

Onion Soup Recipe for 8 Persons

8 onions

1 pint water

4 chicken soup cubes

2 dessertspoons butter

½ pint cream

Version 1

I have only six people to feed and I do not wish to waste any soup.

How would I alter the recipe to give the same type of soup?

Onion soup recipe for 6 persons

.....Onions
.....Water
.....Chicken cubes
.....Butter
.....Cream

Version 2

(a) I am cooking onion soup for 16 people.
How many onions do I need? ....
How much cream do I need? ....

(b) I am cooking onion soup for 4 people.
How much water do I need ....
How many chicken soup cubes
do I need? ....

(c) I am cooking onion soup for 6 people.
How much water do I need? ....
How many chicken soup cubes
do I need? ....
How much cream do I need? ....

Version 3

(a) I am cooking onion soup of 4 people.
How much water do I need? ....
How many chicken soup cubes
do I need? ....

(b) I am cooking onion soup for 6 people.
How much water do I need? ....
How many chicken soup cubes
do I need? ....
How much cream do I need? ....

Three workmen send to the cafe for ham rolls, Peter ate 2, John ate 4, Brian ate 6. The bill came to 36p, how much should each pay?

Peter .... John .... Brian ....

In an office Mr. Adams comes in to work 2 days a week. Mr. Brown comes in to work 4 days a week, Mr. Carter comes in 6 days a week. The bill for making coffee in the office for these three men is 240p. How much should each pay for it to be fair?

Mr. Adams ..... Mr. Brown ..... Mr. Carter .....

In an office Mr. Adams come in to work 2 days a week. Mr. Brown comes in to work 4 days a week. Mr. Carter comes in 6 days a week. The bill for lighting the office for these three men is 240p. How much should each pay for it to be fair?

Mr. Adams ..... Mr. Brown ..... Mr. Carter .....

In a particular chemical compound there are

1 part mercury to 5 parts copper

3 parts tin to 10 parts copper

8 parts zinc to 15 parts copper

State a relationship between the mercury and tin contents and between the zinc and tin.

.... parts mercury to .... parts tin

.... parts zinc to .... parts tin

In a particular metal alloy there are

1 part mercury to 5 parts copper

3 parts tin to 10 parts copper

8 parts zinc to 15 parts copper

You would need how many parts mercury to how many parts tin?

.... parts mercury to .... parts tin

You would need how many parts zinc to how many parts tin?

.... parts zinc to .... parts tin.

4a) There are 3 eels A, B and C in the tank at the Zoo.

15 cm long A

10 cm long B

5 cm long C

The eels are fed sprats, the number depending on their length. If C is fed one sprat, how many sprats should B and A be fed to match?

.... B .... A

If B eats 4 sprats, how many sprats should A and C be fed to match?

.... A C

If A gets 9 sprats, how many sprats should B and C get to match?

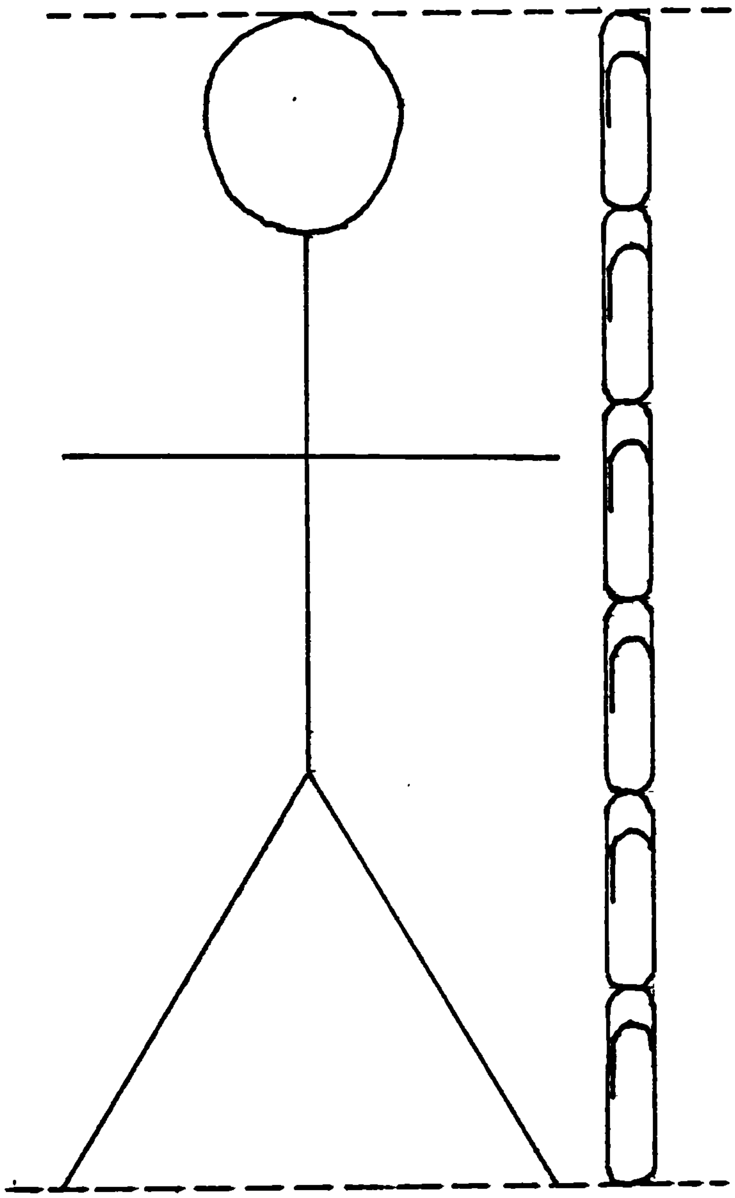
.... B C

4b) As an experiment 3 other eels, X, Y and Z are fed with fish fingers. The length of the fishfinger depending on the length of the eel.

X is 10cm long.
Y is 15cm long.
Z is 25cm long.
If X has a fishfinger 2 cm long, how long should the fish-fingers given to Y and Z be? Y .... Z ....

If Y has a fishfinger 9 cm long, how long should the fish-fingers given to X and Z be? X .... Z ....

If Z has a fishfinger 5 cm long, how long should the fish-fingers given to X and Y be? X .... Y ....



Mr. Short has a friend Mr. Tall. You can see the height of Mr. Short measured with paper clips.

When we measure the height with matchsticks
Mr. Short is four matchsticks tall.
Mr. Tall is six matchsticks tall.
How many paper clips are needed for Mr. Tall's height? ....

a) I am going to Switzerland on holiday. I change £15 into Swiss francs. The rate of exchange is 8 Swiss francs to £1. How many Swiss francs do I get for my £15?

£15 = ...Swiss francs

b) On my return I have 34 Swiss francs left. I change these back into pounds. What do I obtain for them?

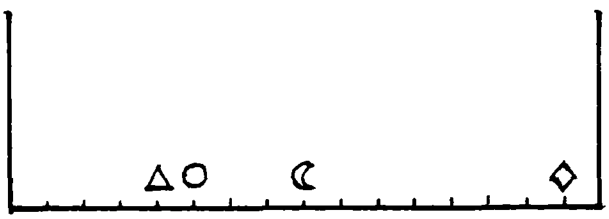
34 Swiss francs = ... £

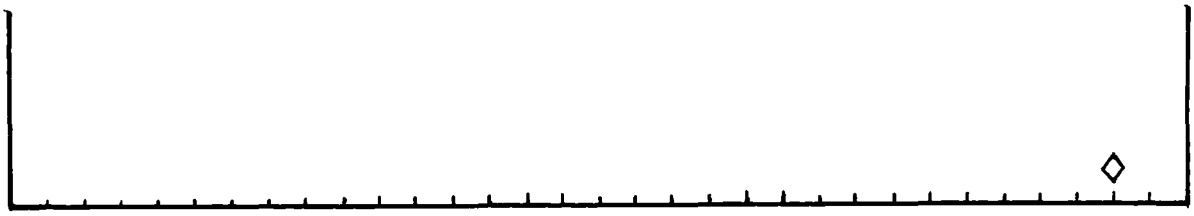
c) If the bank charges me 5% comission to change the money, how much would I get in the two transactions?

£15 =

34 Swiss francs =

Enlarge the small pattern; we have given you the new position of the diamond.





(a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ....

(b) 6% of children in a school have free dinners. There are 250 children in the school.
How many have free dinner? ....

(c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.
What percentage is this? ....

(d) 35 per cent of all adults read a newspaper. If a town has 25,300 adults, how many newspapers would one expect to be sold there?

(i) $\dfrac{35 \times 100}{25{,}300}$ (ii) $\dfrac{25{,}300 \times 35}{100}$ (iii) $\dfrac{25{,}300 \times 100}{35}$

(iv) $\dfrac{35 + 25{,}300}{100}$ (v) $\dfrac{25{,}300}{100 + 35}$ (vi) 35×253

% means <u>per cent</u> or <u>per 100</u>, so 3% is 3 out of every 100.

(a) 4 children out of the hundred on the school trip forgot to bring their lunch.
What percentage is this? ....

(b) 6% of children in a school have free dinners. There are 250 children in the school
How many children have free dinners? ....

(c) The newspaper says that 24 out of 800 Avenger cars have a faulty engine.
What percentage is this? ....

(d) The price of a coat is £20, in the sale it is reduced by 5%; how much does it now cost? ....

APPENDIX 2

Discrimination of Items
(Pilot Study)

CHILD NUMBER (RANKED BY TOTAL SCORE) (1)

Item	Facility	Children 1–10	11–20	21–30	31–40	41–50	51–60
1a(ii)	94	1111111111	1111111111	1111011111	1111111111	1111111011	1111111111
1a(i)	92	1111111111	1111111111	1111111111	1111111111	1111111011	1111111111
3a(i)	87	1111111111	1111111111	1111111110	1111111111	1111111111	1111111111
8 i	84	1111111111	1111111111	1111111111	1111111111	1111111111	1111111111
1b(i)	77	1111111111	1111111111	1111011111	1111111111	1111111001	1111111111
3a(ii)	75	1111111111	1111111111	1111111110	1110101111	1111111101	1110111110
1b(ii)	70	1111111111	1111111111	1110111111	0111110111	1111111001	1111111111
4a(v)	62	1111111111	1111111111	0111111111	1110111011	0111110101	1111011010
2B	51	1110011111	0111110110	1111111111	1110101101	1010000101	0100110111
2A	48	1111011111	0111110110	1111101111	1110111111	1010001101	1111010011
2C	47	1111011111	0111110110	1111101111	1110101111	1110101101	0110010011
3a(iii)	46	1111111110	1001111111	1110111111	0111111111	0001111111	1110111100
3a(iv)	44	1111111101	1111110111	1110111111	0111011111	1111011110	1100111101
3b(i)	43	1111111100	1011111010	1111011011	0110100110	1001110100	1011110101
8 ii	42	1111110011	1100101101	1110011100	1011110000	1111100111	0010010100
4a(iii)	31	1111111111	1101001100	0011100011	1100001011	0111110011	1100001010
4a(ii)	30	1011111111	1101001001	0111101011	1000000101	1011110010	0001001000
6 i	29	1011111110	0111101001	1111101111	0100001100	0100000000	0001100010
8 iii	29	1111101010	0001000101	1101111000	1111001001	0000110111	1101101000
5	25	1111111111	1000010101	0001110000	1100110100	1100000001	0000000000
3b iii	19	1111100011	1010111110	1000000010	1000000101	0100100111	1000000011
3b ii	18	1111111000	1010100011	1000000001	0100000001	1000000111	1011110101
3b iv	18	1111100011	0001110101	0000000010	1000000101	1011000101	0000000011
8 iv	17	1111001100	1010011111	1000000000	0100001001	0001010001	0000001001
1b iii	14	1111111111	0111011010	0100101010	1011111010	0001101010	0000000100
7 ii	9	1100110100	0011001111	0000000000	0100100000	1000001000	0000100000
6 ii	8	1111101000	0011100000	0001001000	0000000000	0000100000	0001000000
7 i	7	1100100100	0100011010	0000000000	0100010000	0000000000	0000000000

All numbers refer to items on final paper.

Question 4b on final paper not included

2 A.B.C marked separately

CHILD NUMBER (RANKED BY TOTAL SCORE) (2)

Item	61–70	71–80	81–90	91–100	101–110	111–120	121–129
1a (ii)	1111111111	1111111111	1111111111	1111111111	1111111111	1111111111	111101111
1a (i)	1111111111	1111111111	1111111111	1111111111	1111111111	1111110111	111101111
3a (i)	1111111111	1111011101	1111111111	1111111111	1101110111	1111111111	111111111
8 (i)	1111101111	1011111101	1110111111	0111110111	0111111111	1111111111	111111111
1b (i)	1111111111	1111111111	1010100001	1111011110	1111101011	1111111101	111111110
3a (ii)	1111101111	1101111011	1011111111	1111011111	1101110111	0101011111	011111111
1b (ii)	1111111101	1111101111	1101111011	1111111110	1101111110	1111101000	111100011
4a (v)	1111111011	1111111101	1101101001	1010111110	1110101001	1000110110	100111000
2B	1100101101	1000000000	0100110001	0010001100	0110001011	0000000100	111000000
2A	1100101101	1101111111	1101001100	0010011001	1001000010	0010001011	111000000
2C	1100101100	1101111111	1101101100	0000110001	1001001010	1010100010	011000000
3a (iii)	0111100010	1101000010	0011011011	0000010000	1110001000	0101000000	010000001
3a (iv)	0100000010	1011010001	0000110011	0000010000	1010010111	1001000000	000011010
3b (i)	0111011011	1111000100	0001101011	1000000010	0001011000	0000100011	010001000
8 ii	1011100010	0011111011	0110011000	1100111001	0101100011	0111001101	000000100
4a (vi)	1000010010	0010101010	0110101001	0111010000	1000000000	0000111000	000010000
4a (HG)	1000000001	0000000001	0011100010	0110110001	0001000000	0001100010	001110000
6 i	0000011111	0010001000	1000001111	0001000010	0100000001	1000111000	001000000
8 iii	1010000000	0011011101	0000001100	1010000001	0001100000	0011010000	000000000
5	0001101100	1000000010	0000000001	1001000000	0000110111	0000001000	000000000
3b iii	0000000100	0000000000	0101010001	1000000000	0000000000	1000000000	000000000
3b ii	0111010000	1000000000	0000000000	0000000000	0000000000	0000000000	010000000
3b iv	0010000010	0000000000	0010101000	1000000000	0100000000	0000000000	000000000
8 iv	0011100000	0010010001	0000001000	0000000000	0010000000	0000000000	001000100
1b iii	0000010000	1000000000	0000000000	0000000000	0100000100	0000000000	000000000
7 ii	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000	010000000
6 ii	0000000000	0000000001	0000000000	0000000000	0000000000	0000000000	000000000
7 i	0000000000	0000000000	0000000000	0000000000	0000000000	0000000000	000000000

CHILD NUMBER (RANKED BY TOTAL SCORE) (2)

Item	130–177
1a.ii	111111111111110111111111111111111011111001000001
1a.i	111111111111111101111111111111111100101110010000
3a(i)	010110011111111110110111101111001111111110001110010000
8(i)	1111101111111011011110111010110100000000011100111010100
1b(i)	0011111111101010111010110010111111011100111000000000000
3a(ii)	0101100111010111101001111011101000111111110000111000100
1b(ii)	1101111000101010001101011100100101000100000000000000
4a(v)	0010010000101010000000001000000101000000100100110000
2B	10111000100000011000000000000000100010101000000010010
2A	101000100000010100000000000000000000001010100000000
2C	1011100100000000010000000000000000000101000000000000
3a(iii)	111000000000011100001000100000000000000000000000000
3a(iv)	010000000000011000010010010000000000000000000000000
3b(i)	00101111100000000110000000000000000000000000000000
8 ii	100010100010000001000000000000000000000000000000000
4a(iii)	0000000000010000000000000000000000000000000000000
4a(iii)	00000100101000000000000000000000000000000000000000
6 i	000001000000000000000100001010000010000000000000000
8 iii	00000100000000000010001000101000000000000000000000
5	0000100010000000000000000100000010000010100000000000
3b iii	0000000000000000000000000000000000000000000000000
3b ii	0000000000000000000001000000000000000000000000000
3b iv	0000000000000000000000000000000000000000000000000
8 iv	0000000000010000000000000000000000000000000000000
1b iii	00000100000000010000000000000000000000000000000000
7 ii	0000000000000000000000000000000000000000000000000
6 ii	0000000000000000000000000000000000000000000000000
7 i	0000000000000000000000000000000100000000000000000

APPENDIX 3

Performance of Children
in Pilot Study

Fig. 1. Comparison of Performance, Three Second Year Classes.

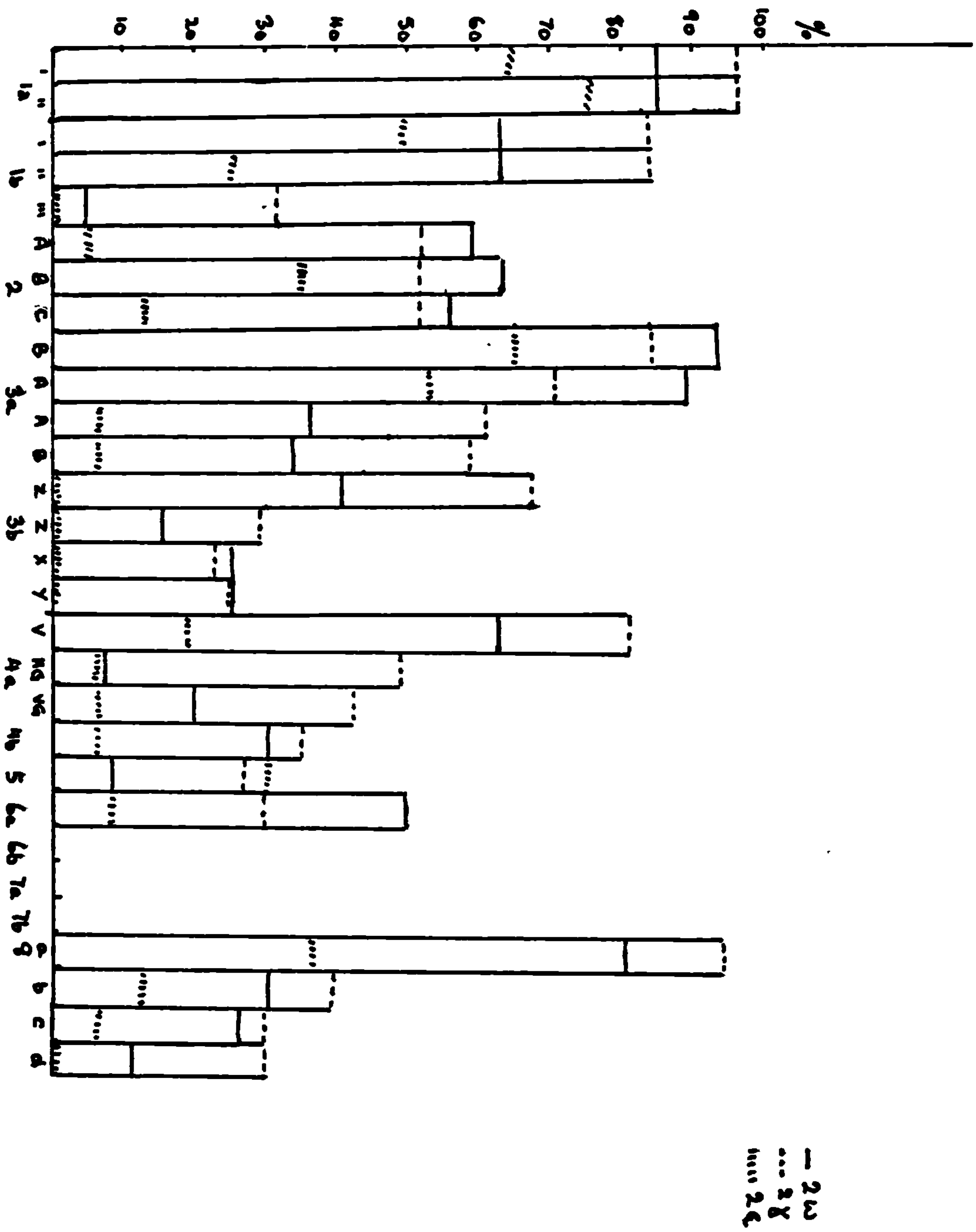


Fig. 2. Comparison of Performance Three Age Groups.

Appendix 4

Sharing Number (From Piaget)

1) Two boys A and B share 30 marbles so that

A has 6 more than B. A has .... B has ....

WORKING

2) Two boys A and B share 43 marbles so that

A has 9 more than B. A has .... B has ....

WORKING

3) Two boys A and B share 5,763 marbles so that A has 574 more than B
How many does each have? Show your working

WORKING

Two boys A and B share x marbles so that A has 14 more than B.

A has .... B has ....

WORKING

APPENDIX 5. Kolmogorov-Smirnov Test (1976 Sample)

The Kolmorov-Smirnov Test for goodness of fit (Siegel 1956) compares the distribution with that of the normal curve. Let $F_o(x)$ be the theoretical frequency distribution function i.e. the distribution obtained from the normal curve.

$S_N(x)$ the observed cumulative frequence distribution

Maximum deviation $D = \text{maximum} \left| F_o(x) - S_N(x) \right|$

2nd Year Data obtained from Appendix 6

IQ≤72	73≤IQ≤77	78≤IQ≤82	83≤IQ≤87	88≤IQ≤92	93≤IQ≤97	98≤IQ≤102
0.0334	.0668	.1216	.2024	.3085	.4338	.5662
0.0258	.0516	.1142	.2014	.3119	.4372	.5650
.0076	.0152	.0074	.0010	-.0034	-.0034	.0012

103≤IQ≤107	108≤IQ≤112	113≤IQ≤117
.6915	.7976	.8784
.6720	.7703	.8637
.0195	.0273	.0147

The last 3 IQ groups IQ 118-128+ give E=98, O=111 no. of children

Largest difference .0273

$$\alpha = 0.05 \qquad D = \frac{1.36}{\sqrt{814}} = \frac{1.36}{28.53} = .0477$$

$$\alpha = 0.01 \qquad D = \frac{1.63}{\sqrt{814}} = .057$$

The distribution approximates to that under the normal curve

APPENDIX 5. (Cont'd)

3rd Year   Data obtained from Appendix 7

$IQ \leq 72$	$73 \leq IQ \leq 77$	$78 \leq IQ \leq 82$	$83 \leq IQ \leq 87$	$88 \leq IQ \leq 92$	$93 \leq IQ \leq 97$	$98 \leq IQ \leq 102$
.0334	.0668	.1216	.2024	.3085	.4338	.5662
.0343	.0699	.1398	.2389	.3418	.4473	.5769
.0009	.0031	.0182	.0365	.0333	.0135	.0107

$103 \leq IQ \leq 107$	$108 \leq IQ \leq 112$	$113 \leq IQ \leq 117$
.6915	.7976	.8784
.6861	.7700	.8628
.0054	.0276	.0156

The last 3 IQ groups
E = 95
0 = 108 no. of children

D = .0365

$\alpha = .05 \qquad D = \frac{1.36}{28.054} = .0485$

$\alpha = .01 \qquad D = \frac{1.63}{28.054} = .0581$

The distribution approximates to the normal distribution.

4th Year   Data obtained from Appendix 8

$IQ \leq 72$	$73 \leq IQ \leq 77$	$78 \leq IQ \leq 82$	$83 \leq IQ \leq 87$	$88 \leq IQ \leq 92$	$93 \leq IQ \leq 97$	$98 \leq IQ \leq 102$
.0334	.0668	.1216	.2024	.3085	.4338	.5662
.0311	.0518	.1037	.1866	.3111	.4320	.5599
.0023	.0150	.0173	.0158	.0026	.0018	.0063

$103 \leq IQ \leq 107$	$108 \leq IQ \leq 112$	$113 \leq IQ \leq 117$
.6915	.7976	.8784
.6959	.7903	.8906
.0044	.0073	.0122

The last 3 IQ groups
E = 105
0 = 95 no. of children

Largest D = .0173

$\alpha = .05$ level $\qquad D = \frac{1.36}{29.462} = .0462$

$\alpha = .01 \qquad D = \frac{1.63}{29.462} = .0553$

Distribution approximates to the normal distribution.

APPENDIX 6

2nd Year Ratio (1976)

Name of School	- 72	75 73-77	80 78-82	85 83-87	90 88-92	95 93-97	100 98-102	105 103-107	110 108-112	115 113-117	120 118-122	125 123-127	128 +
	1976 CSMS Maths Trials							CALVERT results					
(05) 1 class omitted	9	7	21	22	19	30	18	14	11	7	9	3	0
(07) 1 class omitted	0	2	1	11	11	10	13	6	6	9	6	2	0
(08)	2	4	12	9	17	15	20	21	7	7	6	1	0
(01)	1	0	3	2	3	5	3	3	3	2	4	1	0
(16)	9	8	14	27	39	42	39	32	29	29	16	6	2
(17)					1	0	6	4	14	22	23	14	10
(20)							5	7	10				
(22)													8
OBSERVED TOTAL (814)	21 (2.58)	21 (2.58)	51 (6.26)	71 (8,72)	90 (11.05)	102 (12.53)	104 (12.78)	87 (10.7)	80 (9.83)	76 (9.34)	64 (7.86)	27 (3.32)	20 (2.46)
EXPECTED	27	27	44	65	85	101	107	101	85	65	44	27	27
% Ideal Distribution	3.34	3.34	5.48	8.08	10.61	12.53	13.24	12:53	10.61	8.08	5.48	3.34	3.34

APPENDIX 7

3rd Year Ratio (1976)

1976 CSMS Maths Trials — CALVERT results

Name of School	-72	75 73-77	80 78-82	85 83-87	90 88-92	95 93-97	100 98-102	105 103-107	110 108-112	115 113-117	120 118-122	125 123-127	128+
(08)	2	4	12	9	17	15	20	21	7	7	6	1	0
(10)	2	1	6	12	12	11	15	15	10	5	6	7	7
(17)					1	0	6	4	14	22	23	14	10
(18)	3	7	8	18	17	23	26	22	18	24	16	7	1
Minus remedial (19)	10	12	17	20	17	18	8	11	5	4	5	0	0
(04)	10	4	12	19	17	16	27	13	12	11	3	1	1
OBSERVED Total (787)	27	28	55	78	81	83	102	86	66	73	59	30	19
EXPECTED	26.29	26.29	43.13	63.59	83.5	98.6	104.2	98.6	83.5	63.59	43.13	26.29	26.29
% Ideal Distribution	3.34	3.34	5.48	8.08	10.61	12.53	13.24	12.53	10.61	8.08	5.48	3.34	3.34

APPENDIX 8

4th Year Ratio (1976)

1976 CSMS Maths Trials CALVERT results

Name of School	-72	75 73-77	80 78-82	85 83-87	90 88-92	95 93-97	100 98-102	105 103-107	110 108-112	115 113-117	120 118-122	125 123-127	128 +
Half year (06)	1	4	6	8	11	9	12	17	12	9	8	2	1
(11)	5	3	7	12	19	17	24	20	9	8	4	4	3
(14)	11	3	15	19	33	23	25	30	11	17	9	2	0
(01)	1	0	3	2	3	5	3	3	3	2	4	1	0
(15)	0	0	0	4	3	9	8	16	11	17	13	1	6
(16)	9	8	14	27	39	42	39	32	29	29	16	6	2
(22)									7	5	0	3	10
OBSERVED TOTAL (868)	27	18	45	72	108	105	111	118	82	87	54	19	22
EXPECTED	28.99	28.99	47.57	70.13	92.1	108.8	114.9	108.8	92.1	70.13	47.57	28.99	28.99
% Ideal Distribution	3.34	3.34	5.48	8.08	10.61	12.53	13.24	12.53	10.61	8.08	5.48	3.34	3.34

Appendix 9

Letters and Questions to Teachers (1976)

CSMS MATHEMATICS TESTS

1. Has the child intake in your school changed recently or can we assume the 1st, 2nd, 3rd, 4th years draw on approximately the same child population?

2. Is your school split throughout into two matched sections?

3. Which years if any are taught mathematics in mixed-ability classes?

4. Do you set (stream) in Mathematics?

5. Is there any year group you would prefer we did not request for testing?

6. Do you use a particular Mathematics text book or programme?

7. Do you have any I.Q. scores included in you pupil records, to which we could refer if necessary?

SUGGESTED TESTING TIMETABLE

(All test papers supplied and marked by CSMS)

Spring term	-	Calvert Non Verbal Reasoning Tests given to 2nd year.
Easter	-	Specimen papers sent to the school with requests for certain year groups.
Summer Term	-	Testing, papers returned to CSMS before a given date (prior to the vacation).

March 1976

Dear

CSMS is a five-year research project which was set up to investigate children's levels of understanding of concepts in secondary school mathematics and science, and hence help teachers to overcome difficulties in the teaching of these subjects.

Last year the science team of the project (Michael Shayer and Hugh Wylam) worked with your science colleagues and have a great deal of information of some of your pupils as a result of using tests of non-verbal IQ and science tasks. Thus it would be particularly valuable for us in the mathematics team to try our tests on some of your pupils as we would be able to check that our total sample is representative of the total British population of that age by reference to the IQ scores, and we would also be able to make some comparisons between the mathematics and the science tasks. We have already developed several tests, by interviewing children and using trial group testing and are now at the stage where we would like to use three of them to gather information on a larger sample of children. The three tests are

Algebra	(2nd - 5th years)
Ratio	(2nd - 5th years)
Vectors	(3rd - 5th years).

For each year group on each test we will need a total of around 750 children taken across schools of different types.

We would therefore be extremely grateful if you would be willing to arrange for some full year groups in your school to do two of these tests, in order to see how they correlate as measures of mathematics understanding.

We enclose copies of the three tests so that you can have a look through them. We would be asking you to do two of them with your classes, and though we will do our best to give you both your preferences this may not be possible if we have too many schools opting for the same pairs.

We would stress that the tests are designed to ascertain general levels of understanding of a particular topic and not to test teaching effectiveness or the relative efficiency of different school. We hage tried hard to ensure that each test can be used independently of children's previous mathematical experience, so that it would not matter at all if pupils had not met the topic at all.

We hope the tests would also provide useful information for you and your colleagues. We will mark the tests, unless you would particularly like to do so yourselves, and will send you our results, and, later on, a report on the full results from the whole sample. We already have a report on 'number operations' and will send you a copy if you are interested.

A member of the team would bring the tests, and discuss with you any problems, but we would ask you and your staff to conduct the tests yourselves.

If you are interested in helping us perhaps you would return the enclosed sheet to us by April 5th. Please make sure before returning it that the head teacher of the school is agreeable; we will write directly to him if you would like us to do so.

If you have any queriew please phone 01-385 5506 and ask that a member of the maths team phone you, stating the most convenient time for you.

Yours sincerely.

MRS. M. BROWN.

Mathematics Team:	Dr. Kath Hart	(Ratio)
	Dietmar Kuchemann	(Algebra)
	Graham Ruddock	(Vectors)

CSMS

Maths Trials

From

NAME OF TEACHER

SCHOOL

SCHOOL ADDRESS
and
PHONE NUMBER

A Please delete whichever test and year group(s) are not applicable:

I would like copies of the ALGEBRA/RATIO Tests for all 2nd year pupils

I would like copies of the ALGEBRA/RATIO/VECTORS tests for all 3rd year pupils.

I would like copies of the ALGEBRA/RATIO/VECTORS tests for all 4th year pupils

I would like copies of the ALGEBRA/RATIO/VECTORS tests for all 5th year pupils.

There are approximately ..... pupils in each year group.

B. I am unable to try out any of these tests.

Please return by April 5th 1976, using the enclosed envelope.

Instructions for administration of CSMS Tests

Please ask all children to do all their working on the test paper, next to the question preferably, but otherwise on the blank last sheet.

Children should be asked not to use scrap paper or rough books for working.

Vectors

Time required is 1 hour, preferably a double lesson, but if this is not possible two single lessons totalling 1 hour could be used.

The test is designed so that the questions should be answered in the order set, but for weaker children the teacher may wish to suggest that Questions 9 and 10 may be left until last (i.e. after Qu.11 and Qu12). Please note that in Qu11 AE + HG should read $\overrightarrow{AE} + \overrightarrow{HG}$.

The test does not require prior teaching or knowledge of vectors.

No equipment other than a writing implement is necessary, but a ruler is desirable.

Ratio (Test R)

The test should take about 45 minutes. The children need rulers for Question 4.

Graphs

The test should take about 50 min. - 1 hour. Second formers might omit the last two questions. Rulers are needed.

Algebra

The test should take about 40 minutes - please refer to separate sheet for more details.

We would be grateful if completed scripts were sent to us as soon as possible. We will of course refund postage costs and send details of the results when they are available

1976

DATE .... SCHOOL ....

CLASS NAME ....

TEST TAKEN ALG/RATIO/VECTORS/GRAPHS

Have the children in the class covered the topic this year? ....

If 'Yes' what form of teaching material, textbooks etc. was used?

If 'No' is it likely that they have covered it in previous years?

In the teacher's opinion, are there any questions on the test which were too difficult for the class? ....

Vectors test. Are the children familiar with the term "commutativity"?

ONE COPY TO BE COMPLETED FOR EACH TOPIC TEST BY CLASS TEACHER, PLEASE.

APPENDIX 10

Number of Children Taking Ratio Paper (1976)

In ratio the year groups were represented by:-

Year 4		Year 3		Year 2	
Schools	No. of Children	Schools	No. of Children	Schools	No. of Children
01	22	04	141	01	30
06	88			05	177
11	126	08	121	07	79
14	137	10	106	08	119
15	66	17	92	16	277
16	226	18	190	17	89
				22	8
22	25	19	117	20	21
	690		767		800

Appendix 11 Details of the Schools in the 1976 Sample

The location of each school is stated, together with some details of the organisation of the mathematics classes. Although textbooks are quoted, most schools used a number of different texts and supplemented the mathematics work with teacher made worksheets.

01 Mixed Comprehensive in Bristol. The second year uses SMP cards and the older children are taught in sets according to mathematical ability.

04 Mixed Comprehensive in Herts. The first two years are taught as mixed ability classes with a separate remedial group. The last three years are in sets with one remedial group. SMP is used.

05 Boys Comprehensive in Herts. The second year and first year boys are in eight groups, two top and two bottom sets with four of medium ability. The setting is continued higher up the school, SMP numbered series is used.

06 Mixed Comprehensive in Herts. Mixed ability teaching in the first two years, this age group works on the Herts Computer Managed Mathematics Project. The third year is split into three paired sets and after the fourth year the school is halved and each half divided into four ability levels, one being remedial, SMP is used after the second year.

07 Boys Comprehensive in Herts. Only the second year was tested as the rest of the school had been subjected to a different selection procedure. The second year was composed of six sets, one top group, one remedial group and four middle mixed ability groups. The SMP lettered series was used.

08 Mixed Comprehensive in Herts. The first two years were in five sets, a top group, three middle sets and a bottom group. The older children were taught in five groups selected according to ability. SMP lettered series used.

10 Mixed Comprehensive in Herts. The first and second years were taught as mixed ability classes and then the children were placed in sets. SMP lettered series used.

Appendix 11 Contd.

11 Mixed Comprehensive in Herts. Set from the second year onwards, the fourth year contained a class destined for CSE and another for O Level. The books used reflected these two different aims: 'Exercises & Worked Examples in O Level', Clarke; 'CSE Mathematics', Bass & Farnham; 'Metric Mathematics', Raven.

14 Mixed Comprehensive in Glos. Mixed ability teaching in the first two years then the children set for mathematics for the rest of the school. SMP and 'New General Mathematics', Channon & McLeish used.

15 Girls Grammar School, South London. The school had a lower ability intake than would be normal for a grammar school and was about to amalgamate with another school to become comprehensive. The children were set throughout and used SMP.

16 Mixed Comprehensive in Coventry. The first year was taught in mixed ability groups, later year groups were set according to ability. In the fourth year there were two sets destined for O Level, two remedial forms and six other groups. The Scottish Mathematics Group books were used together with other materials.

17 Girls Grammar School in Plymouth. The children were streamed into three classes in each year. Traditional mathematics was taught throughout (Harwood Clarke was the text used for the second years).

18 Mixed Comprehensive in Nottingham. The first two years were taught as mixed ability classes, the third year being split into two parallel bands, each of which were streamed into a top group, two middle groups and two smaller bottom groups. SMP cards were in use.

19 Mixed Comprehensive in Nottingham. Each year group was split into two parallel bands, each band containing three classes chosen by ability with a remedial group in each year. 'Action Mathematics' and SMP cards were used.

20 Middle School in Leeds. Only last year in the school tested, the children were in sets according to ability. Books 7-10 of the Fletcher series were in use.

Appendix 11 contd.

22 Girls Grammar School in Herts. The first year groups were selected according to ability in French, after this year mathematics classes were set according to mathematics ability. The school was the only state grammar school for girls in the area.

Appendix 12

Letters to Teachers 1977

CSMS

May 1977

Dear

We are testing six new Maths tests this summer together with the four we ran last year. I'm writing to ask whether you would be able to help us by running some tests in your school.

Last summer we were able to use four or five schools for each year group in each test. This gave us a fairly representative sample. This year we would again wish to use four or five schools but if we asked that each test be run on an entire year group in each school, the number of scripts to be marked would be beyond our resources, Our solution is to ask that each child in an entire year group does two tests but that in any one class at any one testing time four tests be used. We will do all the assignment of tests to particular children so that a teacher would receive a batch of test papers for a lesson, each test paper being labelled with a child's name. We would therefore ask that you send us the Maths class lists for the year we suggest, if it is convenient for you to test that year. I'm afraid all our new tests require about an hour for completion; this in itself might be inconvenient for you.

Perhaps you would be good enough to let me know:-

1) Can you help us?
2) Is 1 hour (a double period or if necessary two single periods) convenient for a test in your school?
3) Would you mind doing the tests mention on the enclosed sheet?
4) Is the year group suggested convenient? If not, is there another year group which you could offer for testing?
5) Could we have Maths class lists for the year mentioned?

We would hope to let you have the tests mid-June for completion by the end of term.

Yours sincerely

DR. K. HART

SCHOOL:

YEAR:

1st Testing period (1 hour)

1/4 of each class does:

1/4 of each class does:

1/4 of each class does:

1/4 of each class does:

2nd Testing Period (1 hour)

1/4 of each class does:

1/4 of each class does:

1/4 of each class does:

1/4 of each class does:

(We will send test papers labelled if you can send us Maths Class Lists)

Appendix 13

Questionnaire and Instructions for Teachers 1977

ALGEBRA + TEST R	Give out both papers and tell pupils to spend about $\frac{1}{2}$ an hour on each. Half-way through the testing period remind pupils to start on the other test, even if they have not finished the first	Ruler not necessary for ALG. necessary for TEST R

After giving out the papers, it would be very helpful if you would complete the attached questionnaire

School .... Class ....

TEST R (ratio)

Have you covered
enlargement of a figure / centre of enlargement / scale factor
this year?

Has "percentage" been dealt with this year?

Would the children be used to using $\frac{a}{b} = \frac{?}{d}$ to solve ratio problems?

Please give details of any other aspects of ratio that the class has met this year.

Please name or describe any books, etc, that you have used for this purpose:

Appendix 14 Details of the Schools in the 1977 Sample

Some schools used in the 1977 sam_le were the same as those in the 1976 survey, descriptions of these are not repeated and can be found in Appendix 11. Schools common to both 1976 and 1977 testings: 04, 07, 11, 15, 16, 17, 18, 20.

09 Mixed Comprehensive in Herts. The classes were set according to ability from the end of the first year, year two was expected to have completed book D of the SMP lettered series which was used throughout the school although the fourth year had previously been using the MME books.

12 Mixed Comprehensive in Herts. SMP was used throughout the school, which was taught in sets. Focus Mathematics was used in the fourth year as well as SMP.

13 Mixed Comprehensive in Glos. The fourth year was set into two CSE classes, one O Level class and a bottom group which was either to take a CSE Arithmetic examination in the fifth year or no examination. The textbook series by Channon and McLeish was used.

25 Girls Comprehensive in Leeds. Mixed ability teaching took place for the first half term of the first year. The first three years were then set according to ability into two top groups, two middle groups and one bottom set leading in the fourth year to two O Level sets, one CSE mode 1, one CSE mode 3 and the bottom group doing computer studies. The courses were mainly traditional, a variety of books were in use e.g. Channon & McLeish, Parr, New Basic Arithmetic (Howlett).

26 Mixed Comprehensive in Somerset. The school was organised in four ability bands, within the top two bands there were three mixed ability classes within the top band and three within the second band for the first two years. The second band being put in sets in the third year. The two bottom bands each contained one class, the lowest being called 'remedial'. The SMP lettered series was used.

Appendix 14 contd.

28 Mixed Comprehensive in Somerset. The school was split into four bands and within each band the mathematics classes were taught as mixed ability groups until the fourth year. In the fourth year the children were put in classes leading to O Level, CSE modes 1 and 3 and a School Leaving Certificate. SMP was used throughout the school.

33 Mixed Comprehensive in Sheffield. The children were put in sets in the second year and SMP was the series of texts used.

APPENDIX 15 Kolmogorov-Smirnov Test (1977 Sample)

2nd Year (Fractions of entire year group were taken) $\frac{1}{4}$ of schools 26, 12, 25, 33, 9
$\frac{1}{2}$ o f schools 11, 20

The table below shows only the differences between the expected and the obtained total in each IQ range.

Sch	Total	<72	73-77	78-82	83-87	88-92	93-97	98-102	103-7	108-12	113-17	118-122	123-127	128 +
26		6	11	11	11	23	22	14	2	-6	-9	5	5	-
12		0	-3	-1	2	3	10	16	16	9	9	7	2	-
25		-3	-6	-11	-12	-14	-21	-25	-24	-12	-1	2	1	-
33		0	3	4	9	5	3	4	3	2	2	3	2	-
9		-1	-2	-5	-5	-5	-2	-1	-1	1	2	1	0	-
	681	2	3	-2	5	12	12	8	-4	-6	3	18	10	-
11		0	-1	-1	0	4	4	10	14	8	5	2	1	-
20		2	0	1	3	6	-3	-1	-1	0	6	1	3	-
	263	2	-1	0	3	10	1	9	13	8	11	3	4	-

Taking double for schools 11, 20, and adding:-

Total	1207	6	1	-2	11	32	14	26	22	10	25	24	18	-

Quarter Total = 302)
Quarter Greatest Difference 6.5) $\frac{6.5}{302} = 0.0215$

$\alpha = 0.05 \qquad D = \frac{1.36}{\sqrt{302}} = \frac{1.36}{17.38} = .0783$

$\alpha = 0.01 \qquad D = .092$

3rd Year (Fractions of entire year group)

$\frac{1}{4}$ of schools 14, 18, 16
$\frac{1}{2}$ of schools 15, 7
1/20 of schools (Grammar) 17

Taking the appropriate fraction of totals to enable division by 4 at the end:-

Sch	Fr.	Total	<72	73-77	78-82	83-87	88-92	93-97	98-102	103-7	108-12	113-17	118-122	123-127	128 +
14	$\frac{1}{4}$	198	4	1	5	8	20	18	7	22	12	13	11	6	-
16	$\frac{1}{4}$	292	-1	-3	-5	-1	7	12	-12	8	6	11	11	7	-
18	$\frac{1}{4}$	190	-3	-3	-5	-2	-6	-6	-6	-8	-10	-1	5	5	-
15	2/4	176	-6	-12	-22	-28	-40	-44	-52	-21	-38	-18	-2	-6	-
7	2/4	208	20	22	18	30	40	34	32	18	8	10	10	6	-
17	1/20	19	-1	-1	-2	-4	-6	-8	-9	-11	-10	-7	-4	-1	-
		1083	13	4	-11	3	15	6	-16	-13	-32	8	31	17	-

Quarter Total = 271

Quarter Greatest Difference -8

Giving $\frac{8}{271} = .0295$ $\alpha = 0.05$ D = 0.0826

<u>4th Year</u> (Fractions of entire year groups)

$\frac{1}{4}$ schools 28, 13, 25, 4
$\frac{1}{2}$ school 12

Doubling the entry for school 12 to enable division by 4 at the end we obtain:-

Sch	Total	<72	73-77	78-82	83±87	88-92	93-97	98-102	103-7	108-12	113-17	118-122	123-127	128 +
12	322	0	-6	-2	4	6	10	32	32	18	18	14	4	-
13	100	3	2	0	0	-2	0	9	16	15	9	5	3	-
28	197	-5	-4	-5	-9	-5	0	-7	-9	-8	-3	-5	2	-
25	118	-3	-6	-11	-12	-14	-21	-25	-24	-12	-1	2	1	-
04	146	5	4	8	15	17	15	22	7	13	13	8	4	-
	883	0	-10	-10	-2	2	14	31	22	26	36	24	14	-

Quarter Total = 221

Quarter Greatest Difference 9

Giving $\frac{9}{221} = 0.0407$

$\alpha = 0.05 \quad D = 0.0915$

Appendix 16. Piaget Tasks

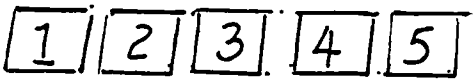
1a We have THREE piles of cards.
One pile has a 1 printed on each card,
one pile has a 2 printed on each card, and
one pile has a 3 printed on each card.

A card is taken from ANY of the piles, and
another card is taken from ANY of the piles;
these two cards are then arranged next to each other, like this.

Find out HOW MANY different arrangements you could make: ....

1b Now we have FOUR piles of cards.
Again, two cards are arranged next to each other, like this.

Find out HOW MANY different arrangements you could make: ....



1c Now we have FIVE piles of cards.
Again, two cards are arranged next to each other, like this.

HOW MANY different arrangements could you make? ....

1d Now we have NINE piles of cards.
Again, two cards are arranged next to each other, like this.

HOW MANY different arrangements could you make? ....

Explain your answer: ....

....

2 Make an accurate copy of these diagrams. Draw your answers on the BACK PAGE of the test paper.

Use a cm ruler only, and WRITE DOWN the length of everything you measure.

Diagram A

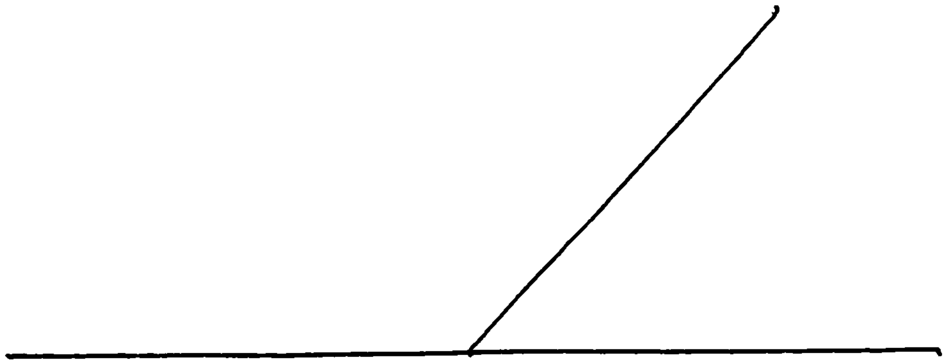


Diagram B

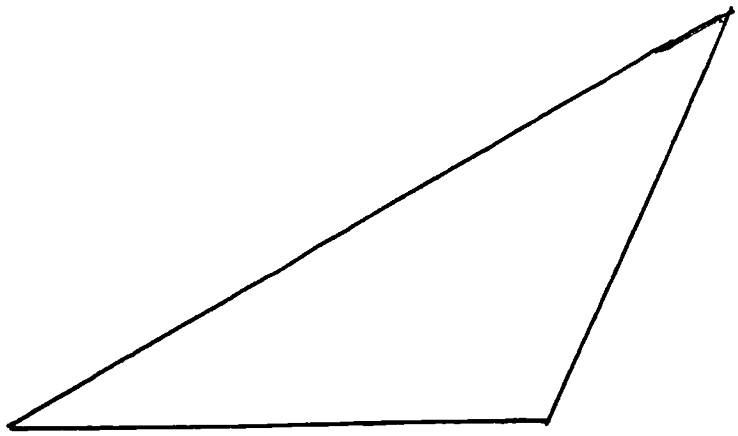
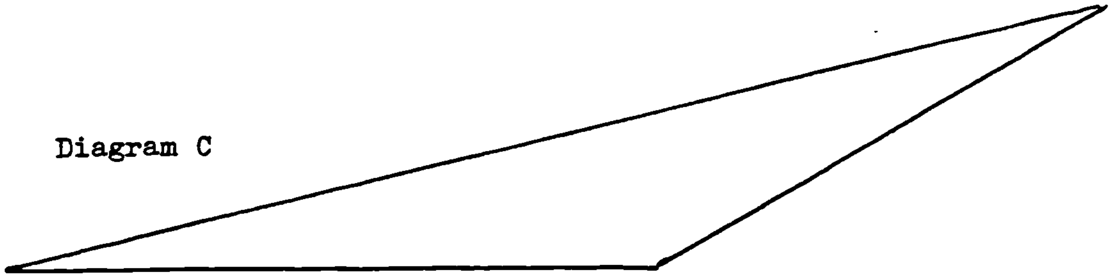
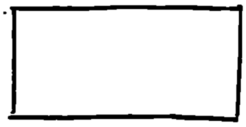


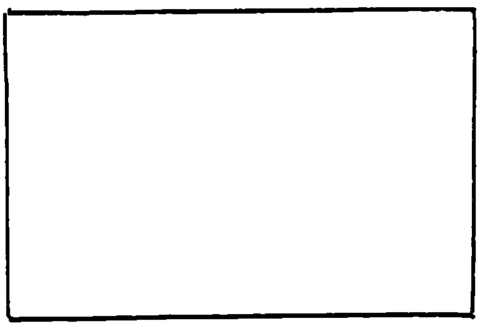
Diagram C



3a. Draw a rectangle exactly the same shape as this rectangle but make your rectangle larger. Use a cm. ruler, and WRITE DOWN the length of everything you measure.

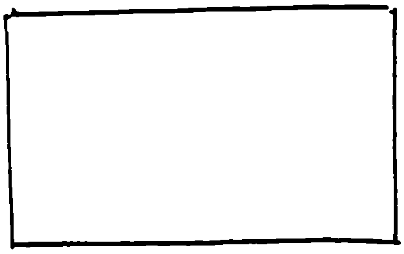


3b. Draw a rectangle exactly the same shape as this rectangle but make your rectangle larger. We have given you the longest side of your new version. Use a cm. ruler, and WRITE DOWN the length of everything you measure.



3c. Draw a rectangle exactly the same shape as this rectangle but make your rectangle larger. We have given you the longer side of your new version.

Use a cm. ruler, and WRITE DOWN the length of everything you measure.



4a 30 sweets are shared between two boys so that one boy has 6 more than the other.

How many sweets does each boy get? ....

Please remember
to write down
your working:

4b 10 sweets are shared between two boys so that one boy has 4 more than the other.

How many sweets does each boy get? ....

4c 10 sweets are shared between John and Peter so that John has 9 and Peter has 1.

How many more sweets does John have than Peter? ....

4d 16 sweets are shared between two boys so that one boy has 6 more than the other.

How many sweets does each boy get? ....

4e 29 sweets are shared between two boys so that one boy has 7 more than the other.

How many sweets does each boy get? ....

4f 188 sweets are shared between two boys so that one boy has 42 more than the other.

How many sweets does each boy get? ....

4g 753 sweets are shared between two boys so that one boy has 29 more than the other.

How many sweets does each boy get? ....

5a Three coats, one red (R), one yellow (Y) and one blue (B) are hung next to each other on three hooks. (Each hook has one coat on it.)

In how many ways can the coats be arranged? ....

5b Three girls, A, B, C are seated on three chairs in a row.

In how many ways can the girls be seated? ....

5c Four girls, A, B, C, D are seated on four chairs in a row.

In how many ways can the girls be seated? ....

5d Another girl, E, comes along, so there are now five girls seated on five chairs in a row.

In how many ways can the girls be seated? ....

Explain your answer: ....

....

Appendix 17: Extract from Teachers' Guide to Ratio Test

Instructions for marking

The marking scheme on the previous page shows the various codes assigned to incorrect and correct answers; the suggested reasons for the incorrect answers have been discussed previously. The code 1 has been assigned to correct answers so a teacher who wishes to obtain a total score would just total the ones.

Interpretation of the codes to find frequency of errors

A frequency count of each code for the class being tested will provide information on the common errors being committed. A preponderence of a particular code for any one pupil will show a consistent incorrect strategy which should then be investigated with that pupil. For example:

Question	1	2a	2b	3	4a	4b	5	6	7	8
AR	11118	1111	1199	1	111	6	6	88	66	1118
BR	11118	4444	4444	8	133	9	9	00	00	1198

AR can successfully deal with the early questions but resorts to the addition strategy on harder items but not on the (easier) eel questions. BR resorts to "the add on one for bigger" strategy for easy items and makes little attempt at the harder items. A preponderence of 9s should be investigated in that it shows incorrect answers to which we have not assigned codes.

Assessment of levels of Understanding

An overall total score will give the teacher little information about the types of question the child is able to do and those he finds too difficult. We have consequently taken some of the items on the test and formed groups of these at different facilities. The grouping is based on the use of homogeneity coefficients, mathematical descriptions and gaps in the facility range derived from the data obtained from testing 2,257 children. The details of the analysis are in Appendix A. Each group of items is called a 'level' and the child is deemed to be capable of that level of understanding if he attains about two thirds of the items in the group. In the trial testing very few children reached criterion level on a higher group without reaching it also on all previous lower groups. Children who fit this pattern should be assigned the highest group they reach. It is suggested that the teacher assign a level of understanding using the following scheme:

Level	Description	Items
0	Unable to make a coherent attempt at any of the questions	Less than 3/5 on level 1
1	No rate needed or rate given. Multiplication by 2, 3 or taking half	3/5 correct of itema 1a(i) 1a(ii), 1b(i), 2a(i), 2a(ii).
2	Rate easy to find or answer can be obtained by taking an amount then half as much again	3/5 correct of items 3, 2a(iii), 2a(iv). 2b(i), 8c.
3	Rate must be found and is harder to find than above. Fraction operation also in this group	4/6 correct of items 2b(ii), 2b(iii), 2b(iv), 1b(iii), 5, 8d.
4	Must recognise that ratio is needed, the questions are complex in either numbers needed or setting.	3/4 correct of items 4b, 6b, 7a, 7b.

A child who scores 3/5 of the level one items but fails to reach the criterion at level two may be very close to completing the second level. By looking at how close to the criterion mark the child comes, the teacher will be able to assess how close to understanding that level of Ratio the child is. Some children will have a gap in the levels 0-4, this may show an area of knowledge which is deficient and in which they need further experience. The levels are in order of difficulty therefore the type of question level 1 children should be given next are like those in level 2, not level 4.

Percentage of children who achieve each level for the three populations in 1976:

	2nd year	3rd year	4th year
Level 0	7%	7%	3%
Level 1	53%	49%	41%
Level 2	26%	23%	27%
Level 3	9%	12%	14%
Level 4	5%	9%	15%

The majority of children in each year achieve levels 1 and 2, that is they can deal with doubling and halving both when applied once or used to build up an answer by taking an amount and adding half of the same amount. The harder items where the child must recognise that a ratio is needed, find the ratio and then apply it, are successfully solved by a few of the children.

The ability to deal with the ratio 2:1 is a poor indicator of the understanding of the total concept of ratio.